

ETH Zürich

Department of Physics

2D BF Theory

Two-Dimensional BF Theory,
Delta Observables, Correlators and Deformations.

Master Thesis

Master HEP

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Zürich, 07 September 2026

Abstract

This thesis studies two-dimensional Abelian BF theory as a topological conformal field theory. We enlarge the BRST cohomology by including new delta-function observables, compute correlation functions involving these operators, and study their properties.

We then consider two deformations of the theory. The first is a superpotential deformation in the ghost fields, while the second is a new deformation generated by a new delta observable. For both deformations, we study the resulting TCFT structure, the modified BRST cohomology, and compute new correlation functions.

Acknowledgements

I would like to express my sincere gratitude to Professor Pavel Mnev for his invaluable support throughout my Master's thesis, for his constant guidance, and for always making himself available for insightful discussions. As a physicist, I found it a truly rewarding and enjoyable experience to delve into mathematical physics and acquire the necessary background in the field under his supervision.

I would also like to thank Professor Alberto S. Cattaneo for welcoming me into his research group for my Master's thesis and for his availability and support. Finally, I am grateful to Professor Matthias R. Gaberdiel for agreeing to serve as my supervisor at ETH Zürich.

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Chapter 1

Introduction

BF theory is a compelling framework in topological field theory for exploring gauge symmetry, BRST cohomology, correlation functions, and the BV formalism. In two dimensions, the theory is particularly interesting because gauge-fixing reveals a topological conformal field theory structure, characterized by a Q -exact stress-energy tensor. It therefore provides a natural setting in which the local methods of conformal field theory meet the cohomological description of observables through BRST cohomology.

The aim of this thesis is first to study two-dimensional Abelian BF theory, following closely the work of Losev, Mnev, and Youmans [1]. We establish its TCFT structure and study its BRST cohomology. One of the main goals is to enlarge the polynomial BRST cohomology by introducing delta-function observables and to study new correlation functions involving this enlarged algebra of observables using Witten's approach [2]. We also discuss the main properties of Q -closed correlators and recall Witten's descent procedure, through which a Q -closed observable generates a tower of descendant observables.

We then study two deformations of the theory. First, we consider the N -component Abelian BF theory with a superpotential deformation. We then introduce a second deformation generated by a distributional observable. For both deformations, we investigate the TCFT structure of the deformed theory, the effect on the original symmetries, the resulting BRST cohomology, and new correlation functions of Q -closed observables. The thesis is structured as follows. Chapter 2 recalls some elements of two-dimensional conformal field theory, notably the correspondence in two dimensions between conformal structures and complex structures, as well as deformations of conformal structures by Beltrami differentials. The latter will be used in Chapter 7 to compute the stress-energy tensor.

Chapter 3 introduces the main concepts of supergeometry needed to present the BRST formalism and, finally, the Batalin–Vilkovisky formalism.

In Chapter 4, we mainly present Witten's approach to computing correlation functions involving delta-function observables, both when the elementary fields have zero modes and when they do not.

Chapter 5 introduces two-dimensional Abelian BF theory. We provide proofs of some of the results of [1], study its TCFT structure and BRST cohomology, and compute new correlation functions involving delta-function observables. We also recall Witten's descent procedure, which will later be used to deform the theory.

Chapter 6 studies a deformation of the N -component two-dimensional Abelian BF theory obtained by introducing a superpotential in the ghost fields. We compute the deformed BRST differential and BRST current, study the associated Feynman diagrams and the quantum TCFT structure of the theory, and analyze the new cohomology of observables.

Chapter 7 follows a similar strategy for a different deformation generated by a distributional observable involving delta functions. We study the deformed BRST structure, stress-energy tensor, correlation functions, and the resulting open BRST algebra. We then construct its BV completion and analyze some of the new contact terms and singularities appearing in the deformed correlators.

Chapter 2

Two-Dimensional Conformal Field Theory

In this section, we briefly present some key elements of two-dimensional conformal field theory. In particular, we discuss the correspondence between two-dimensional conformal structures and complex structures, as well as deformations of complex structures described by Beltrami differentials. These notions will be used in the final chapter. Familiarity with CFT and differential geometry is assumed. Throughout this chapter, we follow [3], [4], and [5].

2.1 Moduli Space of Conformal Structures

Definition 2.1.1 (Conformally flat metric). [5, Definition 2.34] Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) . (M, g) is called *conformally flat* if every point of M has a coordinate neighbourhood U_α , with local coordinates $(x_\alpha^1, \dots, x_\alpha^{p+q})$, in which the metric can be written as

$$g|_{U_\alpha} = \Omega_\alpha^2(x) \left[\sum_{i=1}^p (dx_\alpha^i)^2 - \sum_{j=p+1}^{p+q} (dx_\alpha^j)^2 \right], \quad (2.1)$$

where Ω_α is a smooth and strictly positive function on U_α . Thus, in each such neighbourhood, g differs from the flat metric of signature (p, q) only by the position-dependent conformal factor Ω_α^2 . Coordinates satisfying equation (2.1) are called *conformal coordinates*. In the two-dimensional Riemannian case, they are also commonly referred to as *isothermal coordinates*.

Theorem 2.1.1 (Local conformal flatness in two dimensions). [3]/[5] *Every two-dimensional Riemannian manifold is conformally flat.*

Definition 2.1.2 (Moduli space of conformal structures). [5, p. 47] Let $\text{Conf}(M)$ denote the space of conformal structures on M . The moduli space of conformal structures on M is the quotient of $\text{Conf}(M)$ by the action of the diffeomorphism group $\text{Diff}(M)$:

$$\mathcal{M}_{\text{conf}}(M) = \frac{\text{Conf}(M)}{\text{Diff}(M)}. \quad (2.2)$$

2.2 Complex Structures and Almost Complex Structures

Definition 2.2.1. [4, Definition 8.5] Let M be a differentiable manifold. The pair (M, J) is called an *almost complex manifold* if there exists a tensor J of type $(1, 1)$ such that, for every $x \in M$,

$$J_x^2 = -\text{id}_{T_x M}. \quad (2.3)$$

The tensor J is called an *almost complex structure* on M .

Equation (2.3) implies that the eigenvalues of J_x are $+i$ and $-i$. Since J_x is real, the multiplicity of i is equal to that of $-i$. Hence, M is even-dimensional, and we define its complex dimension by

$$\dim_{\mathbb{C}} M = \frac{1}{2} \dim_{\mathbb{R}} M = n. \quad (2.4)$$

We can extend J_x by complex linearity to the complexified tangent space

$$T_x M^{\mathbb{C}} := T_x M \otimes_{\mathbb{R}} \mathbb{C}, \quad (2.5)$$

while preserving the relation

$$J_x^2 = -\text{id}_{T_x M^{\mathbb{C}}}. \quad (2.6)$$

In this case, following [5, Section 2.8.1], we can split $T_x M^{\mathbb{C}}$ into:

$$T_x M^{\mathbb{C}} = T_x^{1,0} M \oplus T_x^{0,1} M, \quad (2.7)$$

where

$$T_x^{1,0} M = \{X \in T_x M^{\mathbb{C}} \mid J_x X = iX\}, \quad (2.8)$$

$$T_x^{0,1} M = \{X \in T_x M^{\mathbb{C}} \mid J_x X = -iX\}. \quad (2.9)$$

This splitting induces a splitting of the complexified tangent bundle and of the

complexified cotangent bundle:

$$TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M, \quad (2.10)$$

and

$$T^*M^{\mathbb{C}} = T^{*1,0}M \oplus T^{*0,1}M, \quad (2.11)$$

where

$$T^*M^{\mathbb{C}} := T^*M \otimes_{\mathbb{R}} \mathbb{C}. \quad (2.12)$$

More generally, the decomposition (2.11) induces a decomposition of complex-valued differential forms. We define

$$\Omega^{p,q}(M) = \Gamma \left(\bigwedge^p T^{*1,0}M \otimes \bigwedge^q T^{*0,1}M \right). \quad (2.13)$$

An element of $\Omega^{p,q}(M)$ is called a differential form of type (p, q) . The space of complex-valued k -forms then decomposes as

$$\Omega_{\mathbb{C}}^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M). \quad (2.14)$$

Definition 2.2.2 ([5, Definition 2.39]). Let J be an almost complex structure on a smooth manifold M . The almost complex structure J is called *integrable* if there exists an atlas of coordinate charts

$$\{(U_{\alpha}, \varphi_{\alpha})\}, \quad \varphi_{\alpha} = (z^1, \dots, z^n),$$

such that, on each U_{α} ,

$$J \left(\frac{\partial}{\partial z^i} \right) = i \frac{\partial}{\partial z^i}, \quad J \left(\frac{\partial}{\partial \bar{z}^i} \right) = -i \frac{\partial}{\partial \bar{z}^i}.$$

For any two charts $(U_{\alpha}, \varphi_{\alpha})$ and $(U_{\beta}, \varphi_{\beta})$ such that $U_{\alpha} \cap U_{\beta} \neq \emptyset$, the transition map

$$\varphi_{\beta} \circ \varphi_{\alpha}^{-1} : \varphi_{\alpha}(U_{\alpha} \cap U_{\beta}) \longrightarrow \varphi_{\beta}(U_{\alpha} \cap U_{\beta})$$

is holomorphic.

An integrable almost complex structure J is called a *complex structure* on M .

Theorem 2.2.1 (Newlander–Nirenberg). [4, Theorem 8.13] Let (M, J) be an almost complex manifold. Define the Nijenhuis tensor

$$N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

by

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Then the following statements are equivalent [5, Section 2]:

1. The almost complex structure J is integrable.
2. The Nijenhuis tensor vanishes identically:

$$N_J(X, Y) = 0, \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

3. The Dolbeault operator satisfies

$$\bar{\partial}^2 = 0.$$

In real dimension two, any almost complex structure (Σ, J) on a two-dimensional manifold Σ is automatically a complex structure.

Remark. An almost complex structure J naturally induces an orientation on M . Indeed, if $\dim_{\mathbb{R}} M = 2n$ and

$$(e_1, \dots, e_n)$$

is any collection of tangent vectors such that

$$(e_1, Je_1, \dots, e_n, Je_n)$$

forms a basis of $T_x M$, then this basis is declared to be positively oriented. This orientation is independent of the chosen basis.

Theorem 2.2.2 (Equivalence of conformal and complex structures in dimension two). [5, Lemma 2.41] *Let Σ be an oriented smooth manifold of real dimension two. Then there is a one-to-one correspondence between*

- conformal classes of Riemannian metrics on Σ , and
- complex structures on Σ compatible with the given orientation.

More precisely, every conformal class $[g]$ determines a unique compatible complex structure J , and conversely every complex structure J compatible with the orientation determines a unique conformal class of Riemannian metrics.

Definition 2.2.3 (Riemann surface). A *Riemann surface* is a smooth manifold Σ of real dimension two equipped with a complex structure.

2.3 Deformation of the Complex Structure

In our 2D BF theory, we will need to compute the stress-energy tensor, defined by varying the action with respect to the metric. In two dimensions, Theorem 2.2.2 identifies the conformal class of the metric with the compatible complex structure, so variations of the conformal structure can equivalently be described in terms of deformations of the complex structure.

If (M, J) is a complex manifold, deformations of the complex structure can be described, following [5, Section 2.8.3], by deforming the Dolbeault operator $\bar{\partial}$ through a Beltrami differential

$$\mu \in \Omega^{0,1}(M, T^{1,0}M),$$

such that

$$\bar{\partial} \longrightarrow \bar{\partial} + \mu \equiv \bar{\partial}_\mu.$$

The holomorphic Dolbeault operator changes as

$$\partial \longrightarrow \partial + \bar{\mu} \equiv \partial_\mu.$$

Locally, the antiholomorphic Dolbeault operator is written as

$$\bar{\partial}_\mu = d\bar{z}^i \left(\frac{\partial}{\partial \bar{z}^i} + \mu_i^j(z, \bar{z}) \frac{\partial}{\partial \bar{z}^j} \right).$$

The holomorphic Dolbeault operator can be written as

$$\partial_\mu = dz^i \left(\frac{\partial}{\partial z^i} + \bar{\mu}_i^j(z, \bar{z}) \frac{\partial}{\partial \bar{z}^j} \right).$$

To first order in μ and $\bar{\mu}$, the complex structure J changes as

$$J_x \longrightarrow J_x + 2i(\mu_x - \bar{\mu}_x), \quad x \in M.$$

By the Newlander–Nirenberg theorem 2.2.1, for this deformation to define a complex structure, it must satisfy the Kodaira–Spencer equation:

$$(\bar{\partial}_\mu)^2 = 0 \quad \Longleftrightarrow \quad \bar{\partial}_\mu \mu - \frac{1}{2}[\mu, \mu] = 0.$$

Taking the complex conjugate, one obtains

$$\partial \bar{\mu} - \frac{1}{2}[\bar{\mu}, \bar{\mu}] = 0.$$

Chapter 3

Supergeometry and BRST Formalism

The foundational principle of supergeometry is to encode the geometry of spaces within their algebra of functions. Let M be a differentiable manifold. The space of smooth functions $C^\infty(M)$ forms a commutative algebra. For a fixed point $p \in M$, the set of all smooth functions vanishing at p ,

$$\mathfrak{m}_p = \{f \in C^\infty(M) \mid f(p) = 0\},$$

forms a maximal ideal. Indeed, $\mathfrak{m}_p$ is the kernel of the evaluation map

$$\mathrm{ev}_p : C^\infty(M) \longrightarrow \mathbb{R}, \quad f \longmapsto f(p),$$

and

$$C^\infty(M)/\mathfrak{m}_p \simeq \mathbb{R}.$$

Consequently, each point $p \in M$ determines a maximal ideal $\mathfrak{m}_p$.

In supergeometry, this commutative algebra of functions is replaced by a supercommutative algebra. This algebraic generalization leads to the concept of supermanifolds, which generalize ordinary differentiable manifolds by introducing odd (anticommuting) coordinates alongside the usual even coordinates.

Throughout this chapter, we rely on the following primary references: for superlinear algebra, we follow [6, 7]; for the general theory of supermanifolds, we refer to [8, 9]; and for symplectic graded manifolds, as well as the BRST and Batalin–Vilkovisky (BV) formalisms, we primarily follow [9]. For comprehensive, self-contained treatments of these foundational topics, the reader may consult [10, 11].

3.1 Super Linear Algebra

Definition 3.1.1 ($\mathbb{Z}$ -graded vector space). A graded vector space $V^\bullet$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ which decomposes as

$$V^\bullet = \bigoplus_{k \in \mathbb{Z}} V^k.$$

Every nonzero element $v \in V^k$ is called homogeneous of degree k , denoted by

$$|v| = k.$$

Remark. [7, pp. 1.4.7–1.4.8]

1. If $V^\bullet$ is a graded vector space, its dual $V^{\bullet*}$ is a graded vector space defined by

$$(V^*)^i := (V^{-i})^*.$$

2. Let $V^\bullet$ be a graded vector space. For $k \in \mathbb{Z}$, the *shifted vector space* $V^\bullet[k]$ is defined by

$$(V[k])^n := V^{n+k}, \quad n \in \mathbb{Z}.$$

Definition 3.1.2 ($\mathbb{Z}$ -graded supercommutative algebra). [7, Definition 1.4.11] A $\mathbb{Z}$ -graded supercommutative algebra $A^\bullet$ is a $\mathbb{Z}$ -graded vector space

$$A^\bullet = \bigoplus_{k \in \mathbb{Z}} A^k,$$

equipped with a bilinear product

$$A^\bullet \times A^\bullet \longrightarrow A^\bullet, \quad (x, y) \longmapsto xy,$$

such that, for homogeneous elements $x, y \in A^\bullet$,

$$|xy| = |x| + |y|.$$

The graded algebra $A^\bullet$ is required to be:

- *associative*:

$$(xy)z = x(yz),$$

for all homogeneous $x, y, z \in A^\bullet$;

- *graded commutative*:

$$xy = (-1)^{|x||y|}yx,$$

for all homogeneous $x, y \in A^\bullet$.

A graded algebra is called *graded skew-commutative* if

$$xy = -(-1)^{|x||y|}yx$$

for all homogeneous $x, y \in A^\bullet$.

Definition 3.1.3 ($\mathbb{Z}$ -graded Lie algebra). A $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g}^\bullet$ is a $\mathbb{Z}$ -graded vector space

$$\mathfrak{g}^\bullet = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k,$$

equipped with a bilinear bracket

$$[-, -] : \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \longrightarrow \mathfrak{g}^\bullet,$$

such that, for homogeneous elements $x, y, z \in \mathfrak{g}^\bullet$,

1. *Compatibility with the grading*:

$$|[x, y]| = |x| + |y|.$$

2. *Graded skew-commutativity*:

$$[x, y] = -(-1)^{|x||y|}[y, x].$$

3. *Graded Jacobi identity*:

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]].$$

Definition 3.1.4 (Graded derivation). Let $A^\bullet$ be a $\mathbb{Z}$ -graded algebra. A graded derivation D of degree $|D|$ is a linear map such that, for homogeneous $x, y \in A^\bullet$,

$$|D(x)| = |D| + |x|, \quad D(xy) = D(x)y + (-1)^{|D||x|}xD(y) \quad (\text{graded Leibniz rule}).$$

3.2 Supermanifolds

Definition 3.2.1 (Super graded manifold). [8] A smooth $\mathbb{Z}$ -graded supermanifold $\mathcal{M}$ is a locally ringed space

$$\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}}),$$

where M is an ordinary smooth manifold, called the body, and $\mathcal{O}_{\mathcal{M}}$ is a sheaf [10] of $\mathbb{Z}$ -graded supercommutative algebras on M , locally isomorphic to

$$C^\infty(U) \otimes S_{\text{gr}}(W^*),$$

where $U \subset M$ is open and

$$W = \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} W_k$$

is a finite-dimensional $\mathbb{Z}$ -graded real vector space. Here, $S_{\text{gr}}(W^*)$ denotes the graded-symmetric algebra.

The parity of a homogeneous element x is determined by its degree modulo 2:

$$p(x) = |x| \pmod{2},$$

where $|x| = \deg(x) \in \mathbb{Z}$ denotes the degree of x . The local isomorphisms and coordinate transformations are required to preserve the full $\mathbb{Z}$ -grading.

Definition 3.2.2 (Graded vector bundle). A graded vector bundle

$$E_\bullet = \bigoplus_{k \in \mathbb{Z}} E_k \rightarrow M$$

is such that each

$$E_k \rightarrow M$$

is a vector bundle over M .

To obtain a graded manifold with body M in the sense of the previous definition, we assume that $E_0 = 0$. Naturally, the sheaf of sections

$$U \longmapsto \Gamma(U, S_{\text{gr}}(E^*))$$

defines a graded manifold over M . If E is concentrated in odd degrees, then

$$\Gamma(M, S_{\text{gr}}(E^*)) = \Gamma(M, \Lambda^\bullet E^*).$$

Example 3.2.1. [9, Example 4.25] The shifted tangent bundle $T[1]M$ defines a graded manifold with local coordinates x^i of degree 0 and θ^i of degree 1. Its algebra

of functions can be identified with the differential forms on M :

$$C^\infty(T[1]M) \cong \Omega^\bullet(M).$$

Example 3.2.2. [9, Example 4.26] The shifted cotangent bundle $T^*[-1]M$ defines a graded manifold with local coordinates x^i of degree 0 and ψ_i of degree -1 . Its algebra of functions can be identified with the polyvector fields on M with reversed grading:

$$C^\infty(T^*[-1]M) \cong \Gamma(M, \Lambda^\bullet TM).$$

In particular, functions of degree $-k$ correspond to k -vector fields on M .

Definition 3.2.3 (Graded vector fields). A graded vector field X on $\mathcal{M}$ is a graded derivation of $C^\infty(\mathcal{M})$,

$$X : C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M}),$$

in the sense of Definition 3.1.4.

For homogeneous $v, w \in \mathfrak{X}(\mathcal{M})$, the Lie bracket is defined by

$$[v, w] := v \circ w - (-1)^{|v||w|} w \circ v.$$

Definition 3.2.4 (Cohomological vector field). Let $\mathcal{M}$ be a smooth $\mathbb{Z}$ -graded supermanifold. A vector field $Q \in \mathfrak{X}(\mathcal{M})$ is called a *cohomological vector field* if it satisfies:

1. Q has degree 1,

$$|Q| = 1;$$

2. $Q^2 = 0$, or equivalently,

$$[Q, Q] = 0.$$

In this case, the pair

$$(\mathcal{M}, Q)$$

is called a *differential graded manifold* (dg-manifold).

Example 3.2.3. [9, Example 4.30] for the shifted tangent bundle $T[1]M$, the de Rham operator

$$Q = d_M = \sum_i \theta^i \frac{\partial}{\partial x^i}, \quad \theta^i \leftrightarrow dx^i.$$

is a cohomological vector field.

Example 3.2.4. [9, Example 4.31] Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{M} = \mathfrak{g}[1]$ be a graded supermanifold defined by its algebra of functions

$$C^\infty(\mathcal{M}) \cong \Lambda^\bullet \mathfrak{g}^*.$$

If $\{T_a\}$ is a basis of $\mathfrak{g}$ with

$$[T_a, T_b] = f_{ab}^c T_c,$$

and $\{\psi^a\}$ are the corresponding degree-1 coordinates on $\mathfrak{g}[1]$, then

$$Q = d_{\text{CE}} = \frac{1}{2} f_{ab}^c \psi^a \psi^b \frac{\partial}{\partial \psi^c}.$$

This defines a cohomological vector field on $\mathcal{M}$. The condition

$$Q^2 = 0$$

follows from the Jacobi identity on $\mathfrak{g}$.

3.3 Integration on Supermanifolds

Consider a graded vector bundle

$$E \longrightarrow M$$

of rank m , concentrated in odd degrees, where M is an oriented n -dimensional manifold. Following the formulation of [9, Section 4.2.4], it defines a graded manifold $\mathcal{M}$ whose algebra of functions is

$$C^\infty(\mathcal{M}) = \Gamma\left(M, \bigwedge^\bullet E^*\right).$$

We define the Berezinian line bundle of $\mathcal{M}$ by

$$\text{Ber}(\mathcal{M}) = \bigwedge^n T^* M \otimes \bigwedge^m E.$$

Let

$$\mu \in \Gamma(M, \text{Ber}(\mathcal{M}))$$

be a Berezinian. We define an $\mathbb{R}$ -linear integration map on $\mathcal{M}$ by

$$\int_{\mathcal{M}} \mu \cdot : C_c^\infty(\mathcal{M}) \longrightarrow \mathbb{R},$$

where

$$\int_{\mathcal{M}} \mu f := \int_M \langle \mu, (f)_m \rangle. \quad (3.1)$$

In other words, a superfunction $f \in C^\infty(\mathcal{M})$ can be written locally as

$$f = f_0 + f_1 + \cdots + f_m, \quad f_k \in \Gamma\left(M, \bigwedge^k E^*\right).$$

Berezin integration selects the component of highest degree in the odd variables, namely f_m . Here,

$$\langle \cdot, \cdot \rangle : \bigwedge^m E \otimes \bigwedge^m E^* \longrightarrow \mathbb{R}$$

denotes the natural fiberwise pairing. Therefore,

$$\langle \mu, f_m \rangle \in \Omega^n(M)$$

is an ordinary top-degree differential form on M .

The section μ is called a *Berezinian density*. In this construction, μ does not depend on the odd coordinates. More generally, a Berezinian may also depend on the odd coordinates and therefore belongs to

$$\Gamma\left(M, \text{Ber}(\mathcal{M}) \otimes \bigwedge^\bullet E^*\right).$$

Using the algebra of functions on $\mathcal{M}$, this space can equivalently be written as

$$\Gamma\left(M, \text{Ber}(\mathcal{M}) \otimes \bigwedge^\bullet E^*\right) = \Gamma(M, \text{Ber}(\mathcal{M})) \otimes_{C^\infty(M)} C^\infty(\mathcal{M}).$$

Definition 3.3.1 (Divergence of a vector field). [9, Definition 4.45] Let X be a graded vector field on $\mathcal{M}$, and let μ be a Berezinian. The divergence of X with respect to μ , denoted by $\text{div}_\mu(X)$, is defined by

$$\int_{\mathcal{M}} \mu X(f) = - \int_{\mathcal{M}} \mu \text{div}_\mu(X) f,$$

for every

$$f \in C_c^\infty(\mathcal{M}).$$

3.4 Graded Symplectic Manifolds

As seen in Example 3.2.1, for an ordinary smooth manifold M , there is a natural identification

$$C^\infty(T[1]M) \cong \Omega^\bullet(M).$$

Similarly, the algebra of differential forms on a graded manifold $\mathcal{M}$ is defined by

$$\Omega^\bullet(\mathcal{M}) := C^\infty(T[1]\mathcal{M}).$$

Locally, let (x^i, θ^j) be homogeneous coordinates on $\mathcal{M}$, where x^i and θ^j are even and odd, respectively. The induced coordinates on the shifted tangent bundle $T[1]\mathcal{M}$ are

$$(x^i, \theta^j, dx^i, d\theta^j),$$

with degrees

$$|dx^i| = |x^i| + 1, \quad |d\theta^j| = |\theta^j| + 1.$$

The shifted tangent bundle $T[1]\mathcal{M}$ carries the natural cohomological vector field

$$Q = \sum_i dx^i \frac{\partial}{\partial x^i} + \sum_j d\theta^j \frac{\partial}{\partial \theta^j}.$$

Each term in Q has degree 1, and a direct computation gives

$$Q^2 = 0.$$

Under the identification

$$C^\infty(T[1]\mathcal{M}) \cong \Omega^\bullet(\mathcal{M}),$$

the vector field Q corresponds to the de Rham differential d on $\mathcal{M}$.

Definition 3.4.1 (Odd symplectic structure). [9, 12] Let $\omega \in \Omega^2(\mathcal{M})$ be a differential 2-form on $\mathcal{M}$. We call ω an *odd symplectic structure* if the following conditions are satisfied:

1. ω is d -closed:

$$d\omega = 0.$$

2. ω is nondegenerate, i.e., the induced map

$$\begin{aligned} \omega^\flat : T\mathcal{M} &\longrightarrow T^*\mathcal{M}, \\ X &\longmapsto \omega(X, \cdot) \end{aligned}$$

is an isomorphism.

3. ω has odd parity.

In suitable local Darboux coordinates (x^i, θ_i) [12], the odd symplectic form takes

the canonical form

$$\omega = \sum_i dx^i \wedge d\theta_i.$$

The pair $(\mathcal{M}, \omega)$ is called an *odd symplectic supermanifold*.

In the Batalin–Vilkovisky formalism [13], the odd symplectic form is taken to have degree -1 :

$$|\omega| = -1.$$

Definition 3.4.2 (Lagrangian submanifold). A submanifold $\mathcal{L} \subset \mathcal{M}$ is called a *Lagrangian submanifold* if it satisfies the following conditions:

1. $\mathcal{L}$ is isotropic:

$$\omega|_{\mathcal{L}} = 0.$$

2. $\mathcal{L}$ is maximal among isotropic submanifolds.

Following the presentation in [9, p. 90], let $(\mathcal{M}, \omega)$ be an odd symplectic manifold, and let μ be a Berezinian on $\mathcal{M}$. We define the BV Laplacian as the operator

$$\Delta_\mu : C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M}).$$

Let $f \in C^\infty(\mathcal{M})$, and let X_f be the Hamiltonian vector field generated by f , defined by

$$\iota_{X_f} \omega = df. \tag{3.2}$$

The BV Laplacian of f is defined by

$$\Delta_\mu(f) := \frac{1}{2} \operatorname{div}_\mu(X_f). \tag{3.3}$$

We also define the odd Poisson bracket. For every $f, g \in C^\infty(\mathcal{M})$, it is given by

$$\{f, g\} := X_f(g). \tag{3.4}$$

In a local Darboux coordinate chart (x^i, θ^i) , if the Berezinian μ is written locally as

$$\mu = \lambda dx^1 \wedge \cdots \wedge dx^n \wedge d\theta^1 \wedge \cdots \wedge d\theta^n,$$

then the BV Laplacian takes the form

$$\Delta_\mu = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \theta^i} + \frac{1}{2} \{\log \lambda, \cdot\}. \tag{3.5}$$

The odd Poisson bracket is locally given by

$$\{f, g\} = \sum_i f \left(\frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \theta^i} - \frac{\overleftarrow{\partial}}{\partial \theta^i} \frac{\overrightarrow{\partial}}{\partial x^i} \right) g. \quad (3.6)$$

We say that the Berezinian μ is *compatible with* the odd symplectic structure ω if, around every point of $\mathcal{M}$, there exist Darboux coordinates (x^i, θ^i) in which

$$\mu = dx^1 \wedge \cdots \wedge dx^n \wedge d\theta^1 \wedge \cdots \wedge d\theta^n. \quad (3.7)$$

In this case, the BV Laplacian takes the canonical form

$$\Delta_\mu = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \theta^i}. \quad (3.8)$$

The BV Laplacian squares to zero if and only if the Berezinian μ is compatible with ω [9, p. 90]:

$$\Delta_\mu^2 = 0 \quad \Longleftrightarrow \quad \mu \text{ is compatible with } \omega. \quad (3.9)$$

Definition 3.4.3 (BV algebra). [9, Definition 4.69] A *Batalin–Vilkovisky algebra*, or *BV algebra*, is a graded supercommutative unital algebra $(\mathcal{A}, \cdot, 1)$ equipped with the following structures:

- A Poisson bracket of degree -1 ,

$$\{-, -\} : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A},$$

such that, for all homogeneous elements $x, y, z \in \mathcal{A}$,

$$|\{x, y\}| = |x| + |y| - 1,$$

and satisfying

- $\{x, y\} = -(-1)^{(|x|-1)(|y|-1)} \{y, x\};$
- $\{x, yz\} = \{x, y\}z + (-1)^{(|x|-1)|y|} y\{x, z\};$
- $\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|-1)(|y|-1)} \{y, \{x, z\}\}.$

- An $\mathbb{R}$ -linear map of degree -1 , called the *BV Laplacian*,

$$\Delta : \mathcal{A}^j \longrightarrow \mathcal{A}^{j-1},$$

satisfying

- $\Delta^2 = 0$ and $\Delta(1) = 0;$

- $\Delta\{x, y\} = \{\Delta x, y\} + (-1)^{|x|-1}\{x, \Delta y\};$
- The bracket $\{-, -\}$ measures the failure of Δ to satisfy the graded Leibniz rule:

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|}x\Delta y + (-1)^{|x|}\{x, y\}.$$

In this thesis, we adopt the convention in which the BV bracket and the BV Laplacian both have degree -1 , rather than degree $+1$. This convention will be used later to define a BV algebra structure on the space of 0-observables of the two-dimensional BF theory.

3.5 BRST Formalism

In this section, we closely follow the presentation of the BRST formalism in [9].

Definition 3.5.1 (Classical BRST formalism). A classical BRST theory consists of the following data:

1. A differential graded manifold $(\mathcal{F}, Q)$. We call $\mathcal{F}$ the *space of fields* and Q the *BRST operator*.
2. A BRST action $S \in C^\infty(\mathcal{F})$ of degree zero satisfying

$$Q(S) = 0.$$

This expresses the BRST invariance of the action.

The BRST cohomology is defined by

$$H_Q = \frac{\text{Ker } Q}{\text{Im } Q}. \quad (3.10)$$

More precisely, at degree k ,

$$H_Q^k = \frac{\text{Ker}(Q : C^\infty(\mathcal{F})^k \longrightarrow C^\infty(\mathcal{F})^{k+1})}{\text{Im}(Q : C^\infty(\mathcal{F})^{k-1} \longrightarrow C^\infty(\mathcal{F})^k)}.$$

Definition 3.5.2 (Quantum BRST formalism). A quantum BRST theory consists of a classical BRST theory $(\mathcal{F}, Q, S)$ together with a Berezinian μ on $\mathcal{F}$ such that

$$\text{div}_\mu(Q) = 0.$$

For every compactly supported function $f \in C_c^\infty(\mathcal{F})$, we have

$$\int_{\mathcal{F}} \mu Q(f) = - \int_{\mathcal{F}} \mu \text{div}_\mu(Q) f = 0. \quad (3.11)$$

We are generally interested in formal oscillatory path integrals of the form

$$Z = \int_{\mathcal{F}} \mu e^{\frac{i}{\hbar} S}. \quad (3.12)$$

In the presence of gauge redundancy, perturbation theory is generally ill-defined because the critical points of S are degenerate along the gauge orbits. This degeneracy gives rise to zero modes in the Hessian of the action.

To fix this problem, we use Equation (3.11). Let Ψ be a function of degree -1 , called the *gauge-fixing fermion*. Since Q has degree $+1$, the term $Q(\Psi)$ has degree zero and can be added to the action. We then have

$$\int_{\mathcal{F}} \mu e^{\frac{i}{\hbar} S} = \int_{\mathcal{F}} \mu e^{\frac{i}{\hbar} (S + Q(\Psi))}. \quad (3.13)$$

A suitable choice of Ψ removes the degeneracy along the gauge directions and makes the perturbative expansion well-defined.

We define an observable $\mathcal{O}$ in the BRST formalism as a Q -closed superfunction on the space of fields, namely

$$\mathcal{O} \in C^\infty(\mathcal{F}), \quad Q(\mathcal{O}) = 0.$$

The correlation function of such observables $\mathcal{O}_1, \dots, \mathcal{O}_N$ is defined by

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle := \frac{1}{Z} \int_{\mathcal{F}} \mu \mathcal{O}_1 \cdots \mathcal{O}_N e^{\frac{i}{\hbar} S}. \quad (3.14)$$

Equation (3.11) then implies that the correlation function is invariant under the addition of the Q -exact gauge-fixing term:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle = \frac{1}{Z} \int_{\mathcal{F}} \mu \mathcal{O}_1 \cdots \mathcal{O}_N e^{\frac{i}{\hbar} (S + Q(\Psi))}. \quad (3.15)$$

3.6 Classical BV Theory

In the classical BV formalism, the symplectic form has degree -1 , and therefore the induced BV antibracket has degree $+1$. This convention should be distinguished from the degree- -1 bracket that will be used later for the BV algebra of 0-observables in the two-dimensional BF theory.

Definition 3.6.1. [9] A classical BV theory consists of:

1. An odd symplectic manifold $(\mathcal{F}_{\text{BV}}, \omega_{\text{BV}})$, where

$$|\omega_{\text{BV}}| = -1.$$

We call $\mathcal{F}_{\text{BV}}$ the *space of BV fields* and ω_{BV} the *BV 2-form*.

2. A BV action

$$S_{\text{BV}} \in C^\infty(\mathcal{F}_{\text{BV}})$$

of degree zero satisfying the classical master equation

$$\{S_{\text{BV}}, S_{\text{BV}}\} = 0.$$

We then define the BRST operator by

$$Q_{\text{BV}} = \{S_{\text{BV}}, -\}.$$

Since the BV antibracket has degree +1, the operator Q_{BV} also has degree +1. By the graded Jacobi identity and the classical master equation,

$$Q_{\text{BV}}^2(f) = \frac{1}{2}\{\{S_{\text{BV}}, S_{\text{BV}}\}, f\} = 0.$$

Given a classical BRST package $(\mathcal{F}_{\text{BRST}}, Q_{\text{BRST}}, S)$, we can define the corresponding BV package as follows. The space of BV fields is

$$\mathcal{F}_{\text{BV}} = T^*[-1]\mathcal{F}_{\text{BRST}}.$$

If (ϕ^α) are local coordinates on $\mathcal{F}_{\text{BRST}}$, then $\mathcal{F}_{\text{BV}}$ has local coordinates

$$(\phi^\alpha, \phi_\alpha^+),$$

where ϕ_α^+ is the antifield associated with ϕ^α , with degree

$$|\phi_\alpha^+| = -1 - |\phi^\alpha|.$$

The BV symplectic form is

$$\omega_{\text{BV}} = \sum_\alpha \delta\phi^\alpha \wedge \delta\phi_\alpha^+,$$

and the BV action is

$$S_{\text{BV}} = S + \sum_\alpha Q_{\text{BRST}}^\alpha \phi_\alpha^+,$$

where

$$Q_{\text{BRST}} = \sum_\alpha Q_{\text{BRST}}^\alpha \frac{\partial}{\partial \phi^\alpha}.$$

Chapter 4

$\beta\gamma$ System

In this chapter, we investigate the evaluation of correlation functions involving delta-function insertions for a $\beta\gamma$ system, following closely Edward Witten's approach in [2, Section 10].

4.1 The $\beta\gamma$ system

Let Σ be a closed Riemann surface of genus g , and let K_Σ denote its canonical holomorphic line bundle. Consider the fields

$$\gamma \in \Gamma(\Sigma, L), \quad \beta \in \Gamma(\Sigma, K_\Sigma \otimes L^{-1}),$$

where L is a holomorphic line bundle over Σ .

The bosonic $\beta\gamma$ system is defined by the action

$$I_{\beta\gamma} = \frac{1}{4\pi} \int_\Sigma \beta \bar{\partial}_L \gamma. \quad (4.1)$$

We assign the ghost numbers

$$\text{gh}(\beta) = -1, \quad \text{gh}(\gamma) = +1.$$

We denote the kinetic operator by

$$M := \frac{1}{4\pi} \bar{\partial}_L.$$

We also introduce the fermionic system (β^*, γ^*) , with the same geometric properties and opposite statistics, where

$$\gamma^* \in \Gamma(\Sigma, L), \quad \beta^* \in \Gamma(\Sigma, K_\Sigma \otimes L^{-1}).$$

It is defined as well by

$$I_{\beta^* \gamma^*} = \int_{\Sigma} \beta^* M \gamma^*. \quad (4.2)$$

In the absence of zero modes, the bosonic and fermionic Gaussian functional integrals formally give

$$\int \mathcal{D}\beta \mathcal{D}\gamma e^{-I_{\beta\gamma}} = \frac{1}{\det M}, \quad \int \mathcal{D}\beta^* \mathcal{D}\gamma^* e^{-I_{\beta^* \gamma^*}} = \det M.$$

For an observable $\mathcal{O}$, we define the unnormalized correlation function by

$$\langle \mathcal{O} \rangle := \int \mathcal{D}\beta \mathcal{D}\gamma e^{-I_{\beta\gamma}} \mathcal{O}. \quad (4.3)$$

The normalized correlation function is

$$\langle \mathcal{O} \rangle_N := \frac{\langle \mathcal{O} \rangle}{\langle 1 \rangle}. \quad (4.4)$$

4.2 Correlation Functions without Zero Modes

Let

$$G(x, w) := \langle \gamma(x) \beta(w) \rangle_N = \langle \gamma^*(x) \beta^*(w) \rangle_N \quad (4.5)$$

be the inverse of M^{-1} .

For the bosonic system, Wick's theorem gives

$$\left\langle \prod_{i=1}^m \gamma(x_i) \prod_{j=1}^m \beta(w_j) \right\rangle_N = \sum_{\pi \in S_m} \prod_{i=1}^m G(x_i, w_{\pi(i)}). \quad (4.6)$$

For the fermionic system, the sign of the permutation must be included:

$$\left\langle \prod_{i=1}^m \gamma^*(x_i) \prod_{j=1}^m \beta^*(w_j) \right\rangle_N = \sum_{\pi \in S_m} (-1)^\pi \prod_{i=1}^m G(x_i, w_{\pi(i)}). \quad (4.7)$$

4.3 Delta Function Operators

Throughout this section, we continue to assume that the kinetic operator M has no zero modes.

For fermionic fields, delta-function operators are simply

$$\delta(\gamma^*) = \gamma^*, \quad \delta(\beta^*) = \beta^*.$$

They also admit the integral representations

$$\delta(\gamma^*(x)) = \int d\alpha^* e^{-\alpha^* \gamma^*(x)}, \quad \delta(\beta^*(w)) = \int d\lambda^* e^{-\beta^*(w) \lambda^*},$$

where α^* and λ^* are auxiliary fermionic variables with ghost numbers

$$\text{gh}(\alpha^*) = -1, \quad \text{gh}(\lambda^*) = +1.$$

It follows that

$$\langle \delta(\gamma^*(x)) \delta(\beta^*(w)) \rangle_N = G(x, w). \quad (4.8)$$

To express the unnormalized correlator as a Gaussian integral, we define the extended fields and kinetic operator by

$$\widehat{\beta}^* = \begin{pmatrix} \beta^*(z) & \alpha^* \end{pmatrix}, \quad \widehat{\gamma}^* = \begin{pmatrix} \gamma^*(z') \\ \lambda^* \end{pmatrix}, \quad \widehat{M} = \begin{pmatrix} M(z, z') & \delta^{(2)}(z - w) \\ \delta^{(2)}(x - z') & 0 \end{pmatrix}. \quad (4.9)$$

The extended action is

$$\widehat{I}_{\beta^* \gamma^*} := (\widehat{\beta}^*, \widehat{M} \widehat{\gamma}^*) = I_{\beta^* \gamma^*} + \alpha^* \gamma^*(x) + \beta^*(w) \lambda^*,$$

which yields

$$\begin{aligned} \langle \delta(\gamma^*(x)) \delta(\beta^*(w)) \rangle &= \int \mathcal{D}\beta^* \mathcal{D}\gamma^* d\lambda^* d\alpha^* e^{-\widehat{I}_{\beta^* \gamma^*}} \\ &= \det \widehat{M}. \end{aligned} \quad (4.10)$$

With our chosen ordering convention,

$$\det \widehat{M} = (\det M) G(x, w). \quad (4.11)$$

The same construction applies to the bosonic theory. In this case, the Gaussian integral produces the inverse determinant:

$$\langle \delta(\gamma(x)) \delta(\beta(w)) \rangle = \frac{1}{\det \widehat{M}}.$$

After normalization, following [2, Eq. (10.24)], we obtain then

$$\langle \delta(\gamma(x)) \delta(\beta(w)) \rangle_N = \frac{1}{G(x, w)}. \quad (4.12)$$

More generally, following [2, Eq. (10.28)],

$$\left\langle \prod_{i=1}^m \delta(\gamma(x_i)) \prod_{j=1}^m \delta(\beta(w_j)) \right\rangle_N = \frac{1}{\sum_{\pi \in S_m} (-1)^\pi \prod_{i=1}^m G(x_i, w_{\pi(i)})}. \quad (4.13)$$

The delta-function operators are fermionic and carry ghost numbers opposite to those of the corresponding bosonic fields:

$$\text{gh}(\delta(\gamma)) = -1, \quad \text{gh}(\delta(\beta)) = +1.$$

Another useful OPE involving two delta-function operators is [2, Eq. (10.2)]

$$\delta(\gamma(u_1)) \delta(\gamma(u_2)) \sim -\frac{1}{u_1 - u_2} \delta(\gamma(u_2)) \delta(\partial\gamma(u_2)).$$

4.4 Mixed Correlation Functions

We now consider correlation functions containing both elementary fields and delta-function operators. We begin with the fermionic system and then pass to the bosonic one.

In the fermionic system,

$$\langle \delta(\gamma^*(x)) \delta(\beta^*(w)) \gamma^*(x') \beta^*(w') \rangle = (\det \widehat{M}) \widehat{G}(x', w'),$$

where $\widehat{G}$ is the inverse of the extended kinetic operator. Using Wick's theorem, one finds

$$\widehat{G}(x', w') = \frac{G(x, w)G(x', w') - G(x, w')G(x', w)}{G(x, w)}. \quad (4.14)$$

For the bosonic system, the normalized mixed correlation function, following [2, Eq. (10.37)], is

$$\begin{aligned} & \langle \delta(\gamma(x)) \delta(\beta(w)) \gamma(x') \beta(w') \rangle_N \\ &= \frac{G(x, w)G(x', w') - G(x, w')G(x', w)}{G(x, w)^2}. \end{aligned} \quad (4.15)$$

This yields two useful operator product expansions [2, Eq. (10.37), Eq. (10.38)]:

$$\gamma(x') \delta(\gamma(x)) \sim (x' - x) \partial\gamma(x) \delta(\gamma(x)). \quad (4.16)$$

and

$$\beta(w) \delta(\gamma(x)) \sim \frac{1}{w - x} \delta'(\gamma(x)) \quad (4.17)$$

4.5 The $\beta\gamma$ System with Zero Modes

Let

$$n_\gamma := \dim H^0(\Sigma, L), \quad n_\beta := \dim H^0(\Sigma, K_\Sigma \otimes L^{-1})$$

be the numbers of zero modes of γ and β , respectively. By the Riemann–Roch theorem [14],

$$n_\gamma - n_\beta = 1 - g + \deg L. \quad (4.18)$$

We consider the case

$$n_\gamma = 0, \quad n_\beta = r.$$

Let

$$z_1, \dots, z_r \in H^0(\Sigma, K_\Sigma \otimes L^{-1})$$

be a basis of the β zero modes. Equivalently,

$$M^t z_i = 0, \quad i = 1, \dots, r.$$

Since M has zero modes, $\det M = 0$, and the previous vacuum Gaussian formula is no longer applicable. To saturate the r fermionic zero modes, we must insert r copies of β^* . Fermionic statistics yields

$$\sum_{\pi \in S_r} (-1)^\pi \prod_{i=1}^r z_i(w_{\pi(i)}) = \det(z_i(w_j)).$$

Let

$$M^\perp : (\ker M)^\perp \longrightarrow (\ker M^t)^\perp$$

denote the restriction of M to the nonzero-mode sector. The unnormalized fermionic correlation function is then, following [2, Eq. (10.52)],

$$\left\langle \prod_{j=1}^r \beta^*(w_j) \right\rangle = \det M^\perp \det(z_i(w_j)). \quad (4.19)$$

or as delta operators,

$$\left\langle \prod_{j=1}^r \delta(\beta^*(w_j)) \right\rangle = \det M^\perp \det(z_i(w_j)).$$

For the bosonic system, the corresponding result [2, Eq. (10.54)] is

$$\left\langle \prod_{j=1}^r \delta(\beta(w_j)) \right\rangle = \frac{1}{\det M^\perp \det(z_i(w_j))}. \quad (4.20)$$

4.6 General Correlation Functions

We now consider a correlator containing one γ^* insertion and $r + 1$ insertions of β^* :

$$\left\langle \gamma^*(x) \prod_{j=1}^{r+1} \beta^*(w_j) \right\rangle.$$

As a function of each w_j , this correlator is a holomorphic section of

$$K_\Sigma \otimes L^{-1} \otimes \mathcal{O}(x).$$

Under the assumptions above, this space has dimension $r + 1$ by the Riemann–Roch theorem.

The sections $z_1, \dots, z_r$ provide r linearly independent elements. To complete them to a basis, we introduce a section $G(x, w)$ with a simple pole of residue one at $w = x$. It is defined modulo a linear combination of zero modes:

$$G(x, w) \mapsto G(x, w) + \sum_{i=1}^r \alpha_i(x) z_i(w). \quad (4.21)$$

Define

$$N_{(r+1)}(x; \mathbf{w}) := \begin{pmatrix} z_1(w_1) & z_1(w_2) & \cdots & z_1(w_{r+1}) \\ z_2(w_1) & z_2(w_2) & \cdots & z_2(w_{r+1}) \\ \vdots & \vdots & \ddots & \vdots \\ z_r(w_1) & z_r(w_2) & \cdots & z_r(w_{r+1}) \\ G(x, w_1) & G(x, w_2) & \cdots & G(x, w_{r+1}) \end{pmatrix}. \quad (4.22)$$

The unnormalized fermionic correlation function is

$$\left\langle \gamma^*(x) \prod_{j=1}^{r+1} \beta^*(w_j) \right\rangle = \det M^\perp \det N_{(r+1)}(x; \mathbf{w}). \quad (4.23)$$

Equivalently, expressing the insertions as delta functions yields

$$\left\langle \delta(\gamma^*(x)) \prod_{j=1}^{r+1} \delta(\beta^*(w_j)) \right\rangle = \det M^\perp \det N_{(r+1)}(x; \mathbf{w}). \quad (4.24)$$

For the bosonic system,

$$\left\langle \delta(\gamma(x)) \prod_{j=1}^{r+1} \delta(\beta(w_j)) \right\rangle = \frac{1}{\det M^\perp \det N_{(r+1)}(x; \mathbf{w})}. \quad (4.25)$$

Dividing the correlator with elementary insertions by the minimal zero-mode-

soaking correlator yields

$$\frac{\left\langle \gamma(x) \prod_{j=1}^r \delta(\beta(w_j)) \beta(w_{r+1}) \right\rangle}{\left\langle \prod_{j=1}^r \delta(\beta(w_j)) \right\rangle} = \frac{\det N_{(r+1)}(x; \mathbf{w})}{\det N_{(r)}(\mathbf{w})}. \quad (4.26)$$

Here,

$$N_{(r)}(\mathbf{w}) := \left(z_i(w_j) \right)_{1 \leq i, j \leq r}.$$

More generally, consider m insertions of $\delta(\gamma)$ and $r + m$ insertions of $\delta(\beta)$. Define

$$N_{(r+m)}(\mathbf{x}; \mathbf{w}) := \begin{pmatrix} z_1(w_1) & \cdots & z_1(w_{r+m}) \\ \vdots & & \vdots \\ z_r(w_1) & \cdots & z_r(w_{r+m}) \\ G(x_1, w_1) & \cdots & G(x_1, w_{r+m}) \\ \vdots & & \vdots \\ G(x_m, w_1) & \cdots & G(x_m, w_{r+m}) \end{pmatrix}. \quad (4.27)$$

Following [2, Eq. (10.71)], we have

$$\left\langle \prod_{i=1}^m \delta(\gamma(x_i)) \prod_{j=1}^{r+m} \delta(\beta(w_j)) \right\rangle = \frac{1}{\det M^\perp} \frac{1}{\det N_{(r+m)}(\mathbf{x}; \mathbf{w})}. \quad (4.28)$$

The normalized correlation function in the presence of these delta-function insertions is, following [2, Eq. (10.73)],

$$\frac{\left\langle \gamma(x_{m+1}) \prod_{i=1}^m \delta(\gamma(x_i)) \prod_{j=1}^{r+m} \delta(\beta(w_j)) \beta(w_{r+m+1}) \right\rangle}{\left\langle \prod_{i=1}^m \delta(\gamma(x_i)) \prod_{j=1}^{r+m} \delta(\beta(w_j)) \right\rangle} = \frac{\det N_{(r+m+1)}}{\det N_{(r+m)}}. \quad (4.29)$$

Chapter 5

Two-Dimensional Abelian BF Theory

In this chapter, we present the two-dimensional Abelian BF theory in Lorenz gauge, following closely the formulation of Mnev, Losev, and Youmans [1], while providing proofs of some of the results used in their analysis. We then extend the usual polynomial observables by considering distributional operators involving delta functions, compute correlation functions containing such insertions, and compare our expressions with some unpublished results of Mnev, Losev, and Youmans.

We also discuss the main properties of Q -closed correlators and recall Witten's descent procedure, through which a Q -closed observable generates a tower of descendant observables. These observables will play an important role in the following chapters, where we deform the Abelian theory and study how its BRST structure, currents, observables, and correlators are modified with respect to the undeformed theory.

5.1 Topological Conformal Field Theory

Definition 5.1.1 (Topological conformal field theory). [5, Definition 6.9] A topological conformal field theory consists of a space of fields $\mathcal{F}$ endowed with a differential

$$Q : \mathcal{F} \longrightarrow \mathcal{F}$$

of degree $+1$, satisfying the following properties:

1. The holomorphic and antiholomorphic components of the stress-energy tensor are Q -exact:

$$T = Q(G), \quad \bar{T} = Q(\bar{G}).$$

Here, G is a holomorphic primary field of conformal dimension $(2, 0)$ and degree

-1 , while $\bar{G}$ is an antiholomorphic primary field of conformal dimension $(0, 2)$ and degree -1 . The OPEs

$$G(w)G(z), \quad G(w)\bar{G}(z), \quad \bar{G}(w)\bar{G}(z)$$

are regular.

2. There exist two primary fields $J, \bar{J} \in \mathcal{F}$ of degree $+1$ and conformal dimensions $(1, 0)$ and $(0, 1)$, respectively. The differential Q acts on a field $\Phi \in \mathcal{F}$ by

$$Q\Phi(z) = \frac{1}{2\pi i} \oint_{\gamma_z} J(w)\Phi(z) dw + \frac{1}{2\pi i} \oint_{\gamma_z} \bar{J}(\bar{w})\Phi(z) d\bar{w}, \quad (5.1)$$

where γ_z is a small positively oriented contour around z .

3. The total BRST current

$$J_{\text{tot}} = J dz + \bar{J} d\bar{z}$$

satisfies

$$\frac{1}{(2\pi i)^2} \oint_{\gamma_z} \oint_{\gamma_w} J_{\text{tot}}(w) J_{\text{tot}}(z) = 0. \quad (5.2)$$

Hence,

$$Q^2 = 0.$$

5.2 Two-Dimensional Abelian BF Theory in Lorenz Gauge

Let Σ be an oriented closed Riemannian surface. The two-dimensional Abelian BF theory is defined by the classical action

$$S_{\text{BF}} = \int_{\Sigma} B dA, \quad A \in \Omega^1(\Sigma), \quad B \in \Omega^0(\Sigma). \quad (5.3)$$

The action is invariant under the gauge transformations

$$A \longmapsto A + d\epsilon, \quad B \longmapsto B, \quad \epsilon \in \Omega^0(\Sigma). \quad (5.4)$$

We impose the Lorenz gauge condition

$$d \star A = 0.$$

Following the Faddeev–Popov construction [9, Section 4.1], we introduce the Lagrange multiplier λ , together with the Faddeev–Popov ghost fields c and b . The gauge-fixed

action is

$$S = \int_{\Sigma} (B dA + \lambda d*A + b d*dc). \quad (5.5)$$

The corresponding equations of motion are

$$dA = d*A = \Delta b = \Delta c = 0, \quad dB - *d\lambda = 0, \quad (5.6)$$

where $\Delta = *d*d$.

We assign ghost number -1 to b and ghost number $+1$ to c , while A, B, λ have ghost number zero. Thus, the BRST space of fields is

$$\mathcal{F}_{\text{BRST}} = \mathcal{F}_{\text{BRST}}^{-1} \oplus \mathcal{F}_{\text{BRST}}^0 \oplus \mathcal{F}_{\text{BRST}}^1, \quad (5.7)$$

with

$$\mathcal{F}_{\text{BRST}}^{-1} = \Pi\Omega^0(\Sigma)_b, \quad \mathcal{F}_{\text{BRST}}^0 = \Omega^1(\Sigma)_A \oplus \Omega^0(\Sigma)_B \oplus \Omega^0(\Sigma)_\lambda, \quad \mathcal{F}_{\text{BRST}}^1 = \Pi\Omega^0(\Sigma)_c. \quad (5.8)$$

Here, Π denotes parity reversal, so that A, B, λ are even, whereas b, c are odd.

The BRST operator acts as

$$QA = dc, \quad Qb = \lambda, \quad QB = Qc = Q\lambda = 0. \quad (5.9)$$

It has degree $+1$ and satisfies

$$Q : \mathcal{F}_{\text{BRST}}^k \longrightarrow \mathcal{F}_{\text{BRST}}^{k+1}, \quad Q^2 = 0, \quad Q(S) = 0. \quad (5.10)$$

The gauge-fixed action differs from the classical action by a Q -exact term,

$$S = S_{\text{BF}} + Q(\Psi), \quad \Psi = \int_{\Sigma} b d*A, \quad (5.11)$$

where Ψ is the gauge-fixing fermion; see Section 3.5.

5.3 Complex Formulation and Conformal Invariance

By Theorem 2.2.2, the conformal class of the metric on Σ determines a unique compatible complex structure. We may therefore introduce local complex coordinates $(z, \bar{z})$ and use the decomposition introduced in Section 2.2,

$$\Omega^1(\Sigma) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma).$$

The gauge field decomposes locally as

$$A = a dz + \bar{a} d\bar{z}, \quad (5.12)$$

and we combine the real scalar fields B and λ into

$$\gamma = \frac{1}{2}(\lambda + iB), \quad \bar{\gamma} = \frac{1}{2}(\lambda - iB). \quad (5.13)$$

In these coordinates, the gauge-fixed action becomes

$$S = 2i \int_{\Sigma} dz \wedge d\bar{z} \left(\gamma \bar{\partial} a + \bar{\gamma} \partial \bar{a} + b \partial \bar{\partial} c \right). \quad (5.14)$$

In the Koszul convention, the BRST operator acts as

$$Qa = -\partial c, \quad Q\bar{a} = -\bar{\partial} c, \quad Qb = \gamma + \bar{\gamma}, \quad Q\gamma = Q\bar{\gamma} = Qc = 0. \quad (5.15)$$

The metric enters the action (5.14) only through the complex structure determined by its conformal class through $d = \partial + \bar{\partial}$. Then, by Theorem 2.2.2, the gauge-fixed action is invariant under conformal changes of the metric. The two-dimensional Abelian BF theory is therefore classically conformally invariant.

Proposition 5.3.1 (BRST current). *The conserved BRST current is*

$$J_0^{\text{BRST}} = 2i \left(-dz \gamma \partial c + d\bar{z} \bar{\gamma} \bar{\partial} c \right). \quad (5.16)$$

Proof. To derive the BRST current, we introduce a local BRST parameter $\varepsilon = \varepsilon(z, \bar{z})$ and consider the transformations

$$\delta_{\varepsilon} a = -\varepsilon \partial c, \quad \delta_{\varepsilon} \bar{a} = -\varepsilon \bar{\partial} c, \quad \delta_{\varepsilon} b = \varepsilon(\gamma + \bar{\gamma}), \quad \delta_{\varepsilon} c = \delta_{\varepsilon} \gamma = \delta_{\varepsilon} \bar{\gamma} = 0. \quad (5.17)$$

The variation of the action is

$$\begin{aligned} \delta_{\varepsilon} S &= 2i \int_{\Sigma} dz \wedge d\bar{z} \left[\gamma \bar{\partial}(\delta_{\varepsilon} a) + \bar{\gamma} \partial(\delta_{\varepsilon} \bar{a}) + \delta_{\varepsilon} b \partial \bar{\partial} c \right] \\ &= 2i \int_{\Sigma} dz \wedge d\bar{z} \left[-\gamma \bar{\partial}(\varepsilon \partial c) - \bar{\gamma} \partial(\varepsilon \bar{\partial} c) + \varepsilon(\gamma + \bar{\gamma}) \partial \bar{\partial} c \right]. \end{aligned} \quad (5.18)$$

Expanding the derivatives, the terms proportional to ε cancel, yielding

$$\delta_{\varepsilon} S = 2i \int_{\Sigma} dz \wedge d\bar{z} \left(\gamma \partial c \bar{\partial} \varepsilon + \bar{\gamma} \bar{\partial} c \partial \varepsilon \right). \quad (5.19)$$

Using

$$d\varepsilon = dz \partial \varepsilon + d\bar{z} \bar{\partial} \varepsilon,$$

we obtain

$$\delta_\varepsilon S = \int_\Sigma d\varepsilon \wedge J_0^{\text{BRST}},$$

which gives (5.16). □

Writing

$$J_0 = -2i \gamma \partial c, \quad \bar{J}_0 = 2i \bar{\gamma} \bar{\partial} c, \quad (5.20)$$

the total current takes the form

$$J_0^{\text{BRST}} = dz J_0 + d\bar{z} \bar{J}_0.$$

Both components have ghost number +1 and are separately conserved on shell:

$$\bar{\partial} J_0 \underset{\text{EOM}}{\sim} 0, \quad \partial \bar{J}_0 \underset{\text{EOM}}{\sim} 0. \quad (5.21)$$

5.4 Stress-Energy Tensor

The stress-energy tensor $T_{\mu\nu}$ is defined by the variation of the action with respect to the inverse metric [1, Eq. (12)]:

$$\delta_g S = - \int_\Sigma \sqrt{\det g} d^2 x T_{\mu\nu} \delta g^{\mu\nu}. \quad (5.22)$$

From (5.11),

$$S = S_{\text{BF}} + Q(\Psi).$$

Since the classical BF action is metric-independent, the metric dependence of the gauge-fixed action is entirely contained in the gauge-fixing fermion Ψ , through the Hodge star. Following [1], we write

$$\delta_g \Psi = - \int_\Sigma \sqrt{\det g} d^2 x G_{\mu\nu} \delta g^{\mu\nu}.$$

We then obtain

$$\delta_g S = Q(\delta_g \Psi).$$

Hence,

$$T_{\mu\nu} = Q(G_{\mu\nu}), \quad (5.23)$$

so the stress-energy tensor is Q -exact.

Proposition 5.4.1. *The stress-energy tensor of the gauge-fixed Abelian BF theory is Q -exact:*

$$T = Q(G), \quad \bar{T} = Q(\bar{G}),$$

where

$$G = a \partial b, \quad \bar{G} = \bar{a} \bar{\partial} b.$$

The fields G and $\bar{G}$ both have ghost number -1 .

Proof. The gauge-fixing fermion is

$$\Psi = \int_{\Sigma} b d * A.$$

Since Σ is closed, integration by parts gives

$$\Psi = - \int_{\Sigma} db \wedge * A,$$

and hence

$$\delta_g \Psi = - \int_{\Sigma} db \wedge \delta_g (* A). \quad (5.24)$$

Writing $A = A_{\mu} dx^{\mu}$, the Hodge star acts on one-forms as

$$* dx^{\mu} = \sqrt{\det g} g^{\mu\nu} \varepsilon_{\nu\rho} dx^{\rho}. \quad (5.25)$$

Thus,

$$\delta_g (* A) = A_{\mu} \delta_g \left(\sqrt{\det g} g^{\mu\nu} \right) \varepsilon_{\nu\rho} dx^{\rho}.$$

Using

$$\delta_g \sqrt{\det g} = -\frac{1}{2} \sqrt{\det g} g_{\alpha\beta} \delta g^{\alpha\beta},$$

we obtain

$$\begin{aligned} \delta_g \left(\sqrt{\det g} g^{\mu\nu} \right) &= \delta_g \sqrt{\det g} g^{\mu\nu} + \sqrt{\det g} \delta g^{\mu\nu} \\ &= \sqrt{\det g} \left(-\frac{1}{2} g_{\alpha\beta} g^{\mu\nu} + \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu} \right) \delta g^{\alpha\beta}. \end{aligned} \quad (5.26)$$

Substituting into (5.24) yields

$$\delta_g \Psi = - \int_{\Sigma} db \wedge A_{\mu} \sqrt{\det g} \left(-\frac{1}{2} g_{\alpha\beta} g^{\mu\nu} + \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu} \right) \varepsilon_{\nu\rho} dx^{\rho} \delta g^{\alpha\beta}. \quad (5.27)$$

In local complex coordinates $(z, \bar{z})$,

$$A = a dz + \bar{a} d\bar{z}, \quad db = \partial b dz + \bar{\partial} b d\bar{z},$$

and the metric variation becomes

$$\delta_g \Psi = -\frac{i}{2} \int_{\Sigma} dz \wedge d\bar{z} \left(a \partial b \delta g^{zz} + \bar{a} \bar{\partial} b \delta g^{\bar{z}\bar{z}} \right). \quad (5.28)$$

Comparing (5.28) with

$$\delta_g \Psi = -\frac{i}{2} \int_{\Sigma} dz \wedge d\bar{z} G_{\mu\nu} \delta g^{\mu\nu},$$

gives

$$G_{zz} = a \partial b, \quad G_{\bar{z}\bar{z}} = \bar{a} \bar{\partial} b, \quad G_{z\bar{z}} = 0.$$

Defining

$$G = G_{zz} = a \partial b, \quad \bar{G} = G_{\bar{z}\bar{z}} = \bar{a} \bar{\partial} b,$$

we have

$$G_{\text{tot}} = (dz)^2 G + (d\bar{z})^2 \bar{G}. \quad (5.29)$$

□

Since

$$\delta_g S = Q(\delta_g \Psi),$$

comparison with the definition of the stress-energy tensor gives

$$T_{\mu\nu} = Q(G_{\mu\nu}).$$

we obtain then

$$T = T_{zz} = Q(a \partial b) = -\partial c \partial b + a \partial \lambda, \quad (5.30)$$

$$\bar{T} = T_{\bar{z}\bar{z}} = Q(\bar{a} \bar{\partial} b) = -\bar{\partial} c \bar{\partial} b + \bar{a} \bar{\partial} \lambda. \quad (5.31)$$

5.5 Correlators of Elementary Fields

The action S decomposes into three independent free sectors:

$$S = S_{\gamma a} + S_{\bar{\gamma} \bar{a}} + S_{bc}, \quad (5.32)$$

where

$$\begin{aligned} S_{\gamma a} &= 2i \int_{\Sigma} dz \wedge d\bar{z} \gamma \bar{\partial} a, & S_{\bar{\gamma} \bar{a}} &= 2i \int_{\Sigma} dz \wedge d\bar{z} \bar{\gamma} \partial \bar{a}, \\ S_{bc} &= 2i \int_{\Sigma} dz \wedge d\bar{z} b \partial \bar{\partial} c. \end{aligned} \quad (5.33)$$

To compute the correlator

$$\langle a(w) \gamma(z) \rangle,$$

we invert the operator $\bar{\partial}$. On $\mathbb{C}$, we require the fields to vanish at infinity, which removes the zero modes of the Dolbeault operator. The same condition is imposed

on the antiholomorphic system $(\bar{a}, \bar{\gamma})$. We then obtain

$$\langle a(w)\gamma(z) \rangle_{\mathbb{C}} = \frac{1}{w-z}. \quad (5.34) \quad \langle \bar{a}(w)\bar{\gamma}(z) \rangle_{\mathbb{C}} = \frac{1}{\bar{w}-\bar{z}}. \quad (5.35)$$

To compute

$$\langle c(w)b(z) \rangle_{\mathbb{C}},$$

we invert the second-order operator $\Delta = 4\partial\bar{\partial}$. To remove its constant zero mode and regularize the infrared behavior, we replace the complex plane by a disk of large radius R and impose Dirichlet boundary conditions:

$$D_R = \{z \in \mathbb{C} \mid |z| \leq R\}. \quad (5.36) \quad b|_{\partial D_R} = 0, \quad c|_{\partial D_R} = 0. \quad (5.37)$$

For points far from the boundary and in the large- R limit, this gives

$$\langle c(w)b(z) \rangle_{\mathbb{C}} = 2 \log \left(\frac{|w-z|}{R} \right). \quad (5.38)$$

Equations (5.34)–(5.38) are the only independent non-vanishing two-point functions between elementary fields. Since the theory is free, all higher correlation functions follow from them by Wick's theorem.

On $\mathbb{CP}^1$, the scalar fields $\gamma, \bar{\gamma}, b$, and c have constant zero modes. In contrast, a and $\bar{a}$ have no zero modes, since $\mathbb{CP}^1$ admits no nonzero global holomorphic or antiholomorphic one-forms.

Consequently, non-vanishing correlators on $\mathbb{CP}^1$ must absorb all zero modes. The bosonic zero modes of γ and $\bar{\gamma}$ and the fermionic zero modes of b and c can be absorbed by appropriate delta-function insertions (see Chapter 4)

5.6 Composite Fields and Delta Observables

A composite field is a local polynomial in the elementary fields and their derivatives, considered modulo the equations of motion and their derivatives. Following [1, Section 2.2], we define the space of composite fields by

$$\mathbb{F} := \frac{\text{Sym}^\bullet \text{Jet}(\underline{F}_{\mathbb{C}}^*)}{\langle \text{equations of motion} \rangle}, \quad (5.39)$$

where

$$\underline{F}_{\mathbb{C}}^* := \mathbb{C} \otimes \underline{F}^*.$$

A composite field inserted at a point $z \in \Sigma$ is an element of the fiber $\mathbb{F}_z$. The BRST operator induces a differential of degree +1 on $\mathbb{F}_z$, acting as a graded derivation and commuting with spacetime derivatives. The algebra of polynomial local observables

is the corresponding BRST cohomology:

$$\mathcal{O}_z := H_Q^\bullet(\mathbb{F}_z) = \mathbb{C}[B, c], \quad (5.40)$$

where $\mathbb{C}[B, c]$ is understood as a graded polynomial algebra, with c odd.

We enlarge $\mathbb{F}_z$ by allowing the delta operators

$$\delta(a), \quad \delta(\bar{a}), \quad \delta(\gamma), \quad \delta(\bar{\gamma}), \quad \delta(B) \quad \delta(\lambda) \quad \delta(b), \quad \delta(c),$$

together with their distributional derivatives and admissible finite products. We denote the resulting space by

$$\mathbb{F}_z \subset \mathbb{F}_z^\delta.$$

The enlarged space contains additional Q -closed local fields, including

$$\begin{aligned} & \delta(a) \partial c, \quad \delta(\bar{a}) \bar{\partial} c, \quad b \delta(\lambda), \quad \delta(B), \\ & \delta^{(n)}(B), \quad \delta^{(n)}(\gamma), \quad \delta^{(n)}(\bar{\gamma}), \quad b \delta(\gamma) \delta(\bar{\gamma}), \quad n \geq 1. \end{aligned} \quad (5.41)$$

We restrict this extension to Dirac delta distributions and their distributional derivatives, together with their admissible finite products with local fields.

The field $\delta(\lambda)$ is Q -closed but also Q -exact:

$$\delta(\lambda) = -Q(b \delta'(\lambda)). \quad (5.42)$$

More generally,

$$\delta^{(n)}(\lambda) = -\frac{1}{n+1} Q(b \delta^{(n+1)}(\lambda)), \quad n \geq 0. \quad (5.43)$$

We assume that the extension from $\mathbb{F}_z$ to $\mathbb{F}_z^\delta$ preserves the TCFT structure, so that the action of Q and the corresponding descent relations extend consistently to the distributional local fields.

The extended space of local observables is

$$\mathcal{O}_z^\delta := H_Q^\bullet(\mathbb{F}_z^\delta). \quad (5.44)$$

We distinguish the polynomial BRST cohomology $\mathcal{O}_z$ from its distributional extension $\mathcal{O}_z^\delta$.

5.7 Operator Product Expansions

In two-dimensional Abelian BF theory, we obtain the following OPEs [1, section 2.3]:

$$T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg.}, \quad (5.45)$$

$$J(w)G(z) \sim -\frac{1}{(w-z)^3} + \frac{(\gamma a)(z)}{(w-z)^2} + \frac{T(z)}{w-z} + \text{reg.}, \quad (5.46)$$

$$T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.}, \quad (5.47)$$

$$T(w)J(z) \sim \frac{J(z)}{(w-z)^2} + \frac{\partial J(z)}{w-z} + \text{reg.} \quad (5.48)$$

Thus, J is a primary field of conformal dimension $(1, 0)$, while G is a primary field of conformal dimension $(2, 0)$. Similarly, $\bar{J}$ and $\bar{G}$ are primary fields of conformal dimensions $(0, 1)$ and $(0, 2)$, respectively.

We can also verify that the BRST differential is generated by the currents J and $\bar{J}$. For every field Φ , one has

$$Q\Phi(z) = \frac{1}{2\pi i} \oint_{\gamma_z} J(w)\Phi(z) dw + \frac{1}{2\pi i} \oint_{\gamma_z} \bar{J}(\bar{w})\Phi(z) d\bar{w}, \quad (5.49)$$

where γ_z is a small contour around z .

We conclude that Two-dimensional Abelian BF theory is a topological conformal field theory.

5.8 Correlation Functions with Delta Observables

In this section, we evaluate several correlators involving delta-function observables on the complex plane $\mathbb{C}$ and on the sphere $\mathbb{CP}^1$.

Example 5.8.1 (A correlator on $\mathbb{C}$). We compute

$$\langle \delta(a(w)) \partial c(w) b(z) \delta(\gamma(z)) \rangle_{\mathbb{C}}$$

The correlator factorizes as

$$\langle \delta(a(w)) \partial c(w) b(z) \delta(\gamma(z)) \rangle_{\mathbb{C}} = \langle \delta(a(w)) \delta(\gamma(z)) \rangle_{\mathbb{C}} \langle \partial c(w) b(z) \rangle_{\mathbb{C}}. \quad (5.50)$$

Using (4.12), we obtain

$$\langle \delta(a(w)) \delta(\gamma(z)) \rangle_{\mathbb{C}} = w - z, \quad \langle \partial c(w) b(z) \rangle_{\mathbb{C}} = \frac{1}{w - z}. \quad (5.51)$$

Hence,

$$\langle \delta(a(w)) \partial c(w) b(z) \delta(\gamma(z)) \rangle_{\mathbb{C}} = 1. \quad (5.52)$$

Example 5.8.2 (A correlator on $\mathbb{CP}^1$). Let

$$\tilde{\mathcal{O}}(x) := b(x) \delta(\gamma(x)) \delta(\bar{\gamma}(x))$$

We compute

$$\langle a(w) \gamma(z) \tilde{\mathcal{O}}(x) c(y) \rangle_{\mathbb{CP}^1}$$

The correlator factorizes as

$$\langle a(w) \gamma(z) \tilde{\mathcal{O}}(x) c(y) \rangle_{\mathbb{CP}^1} = \langle a(w) \gamma(z) \delta(\gamma(x)) \rangle_{a\gamma, \mathbb{CP}^1} \langle \delta(\bar{\gamma}(x)) \rangle_{\bar{a}\bar{\gamma}, \mathbb{CP}^1} \langle b(x) c(y) \rangle_{bc, \mathbb{CP}^1} \quad (5.53)$$

The correlator $\langle b(x) c(y) \rangle_{bc, \mathbb{CP}^1}$ absorbs the zero modes of b and c , while $\langle \delta(\bar{\gamma}(x)) \rangle_{\bar{a}\bar{\gamma}, \mathbb{CP}^1}$ absorbs the zero mode of $\bar{\gamma}$; both contribute only normalization constants.

Using Witten's formula (4.26), we obtain

$$\frac{\langle a(w) \gamma(z) \delta(\gamma(x)) \rangle_{a\gamma, \mathbb{CP}^1}}{\langle \delta(\gamma(x)) \rangle_{a\gamma, \mathbb{CP}^1}} = \frac{1}{w-x} - \frac{1}{w-z}. \quad (5.54)$$

Up to normalization, we obtain

$$\langle a(w) \gamma(z) \tilde{\mathcal{O}}(x) c(y) \rangle_{\mathbb{CP}^1} = \frac{x-z}{(w-x)(w-z)}. \quad (5.55)$$

As expected from the OPEs, the correlator vanishes for $x = z$ and has poles at $w = z$ and $w = x$.

5.9 Three-Delta Correlators

Example 5.9.1 (The correlator of $\delta(A)$, $\delta(B)$, and $\delta(\lambda)$). We compute

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) \rangle_{\mathbb{C}},$$

where

$$A = a dz + \bar{a} d\bar{z}, \quad B = -i(\gamma - \bar{\gamma}), \quad \lambda = \gamma + \bar{\gamma}.$$

We first replace the bosonic fields with fields $a^*, \bar{a}^*, \gamma^*, \bar{\gamma}^*$ of opposite statistics. Since these fields are fermionic, their delta functions are identified with the fields

themselves. We have then,

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) \rangle_{\text{ferm}} = \frac{i}{|dz_1| |d\bar{z}_1|} \left\langle a^*(z_1) \bar{a}^*(z_1) (\gamma^* - \bar{\gamma}^*)(z_2) (\gamma^* + \bar{\gamma}^*)(z_3) \right\rangle. \quad (5.56)$$

Only the terms containing one holomorphic and one antiholomorphic contraction survive. Using

$$\langle a^*(z_1) \gamma^*(z_2) \rangle = \frac{1}{z_1 - z_2}, \quad \langle \bar{a}^*(z_1) \bar{\gamma}^*(z_2) \rangle = \frac{1}{\bar{z}_1 - \bar{z}_2},$$

we obtain

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) \rangle_{\text{ferm}} = \frac{i}{|dz_1| |d\bar{z}_1|} \left[\frac{1}{(z_1 - z_3)(\bar{z}_1 - \bar{z}_2)} + \frac{1}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_3)} \right]. \quad (5.57)$$

The corresponding bosonic Gaussian integral is the inverse of this fermionic result. Therefore, up to an overall normalization,

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) \rangle_{\mathbb{C}} = \frac{|z_1 - z_2|^2 |z_1 - z_3|^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_3) + (z_1 - z_3)(\bar{z}_1 - \bar{z}_2)}. \quad (5.58)$$

Singular locus of the correlator. The denominator of (5.58) vanishes precisely when

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_3) + (z_1 - z_3)(\bar{z}_1 - \bar{z}_2) = 0. \quad (5.59)$$

Equivalently,

$$\frac{z_1 - z_2}{z_1 - z_3} \in i\mathbb{R}. \quad (5.60)$$

Thus, for fixed z_2 and z_3 , the singular locus of z_1 is the circle whose diameter is the segment $[z_2, z_3]$.

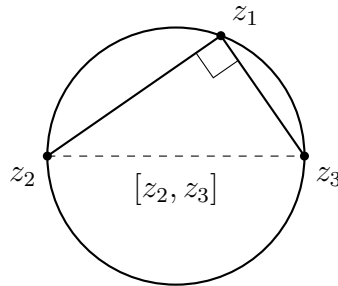


Figure 5.1: For fixed z_2 and z_3 , the denominator vanishes when z_1 lies on the circle with diameter $[z_2, z_3]$.

Remark. The same correlator can be obtained on $\mathbb{CP}^1$ by inserting $b(x)\delta(\gamma(x))\delta(\bar{\gamma}(x))$ together with $c(y)$, thereby absorbing the constant zero modes of $\gamma, \bar{\gamma}, b$, and c . More

precisely,

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) \rangle_{\mathbb{C}} \sim \lim_{x \rightarrow \infty} \langle \delta(A(z_1)) \delta(B(z_2)) \delta(\lambda(z_3)) b(x) \delta(\gamma(x)) \delta(\bar{\gamma}(x)) c(y) \rangle_{\mathbb{CP}^1} \quad (5.61)$$

Example 5.9.2 (The correlator of $\delta(A)$ and two $\delta(B)$ insertions). We compute

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(B(z_3)) \rangle_{\mathbb{C}}.$$

Using the same construction as before, passing first to fields of opposite statistics and then returning to the bosonic theory, we obtain directly, up to normalization,

$$\langle \delta(A(z_1)) \delta(B(z_2)) \delta(B(z_3)) \rangle_{\mathbb{C}} = \frac{|z_1 - z_2|^2 |z_1 - z_3|^2}{(z_1 - z_3)(\bar{z}_1 - \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_3)}. \quad (5.62)$$

The denominator vanishes precisely when

$$\frac{z_1 - z_3}{z_1 - z_2} = \frac{\bar{z}_1 - \bar{z}_3}{\bar{z}_1 - \bar{z}_2} \quad (5.63)$$

Thus, the correlator has a pole if and only if the three points z_1 , z_2 , and z_3 are collinear.

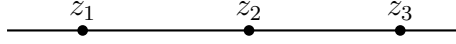


Figure 5.2: The singular configuration in which z_1 , z_2 , and z_3 are collinear.

5.10 Generalized Correlators

Proposition 5.10.1 (General $\delta(a)$ – $\delta(\gamma)$ correlator). *Up to an overall normalization,*

$$\left\langle \prod_{i=1}^n \delta(a(w_i)) \prod_{j=1}^{n+1} \delta(\gamma(z_j)) \right\rangle_{\mathbb{CP}^1} \sim (-1)^n \frac{\prod_{i=1}^n \prod_{j=1}^{n+1} (w_i - z_j)}{\prod_{1 \leq i < j \leq n} (w_j - w_i) \prod_{1 \leq i < j \leq n+1} (z_i - z_j)}. \quad (5.64)$$

Proof. By Witten's formula (5.55),

$$\left\langle \prod_{i=1}^n \delta(a(w_i)) \prod_{j=1}^{n+1} \delta(\gamma(z_j)) \right\rangle_{\mathbb{CP}^1} \sim \frac{1}{\det N_{n+1}}, \quad (5.65)$$

where

$$N_{n+1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{w_1 - z_1} & \frac{1}{w_1 - z_2} & \cdots & \frac{1}{w_1 - z_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{w_n - z_1} & \frac{1}{w_n - z_2} & \cdots & \frac{1}{w_n - z_{n+1}} \end{pmatrix}.$$

The first row can be obtained by introducing an auxiliary point x and taking the limit $x \rightarrow \infty$:

$$\det N_{n+1} = \lim_{x \rightarrow \infty} x \begin{vmatrix} 1 & \cdots & 1 \\ \frac{x - z_1}{1} & \cdots & \frac{x - z_{n+1}}{1} \\ \frac{1}{w_1 - z_1} & \cdots & \frac{1}{w_1 - z_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{w_n - z_1} & \cdots & \frac{1}{w_n - z_{n+1}} \end{vmatrix}. \quad (5.66)$$

The determinant in (5.66) is a Cauchy determinant. Using the Cauchy determinant formula and taking $x \rightarrow \infty$, we obtain

$$\det N_{n+1} = (-1)^n \frac{\prod_{1 \leq i < j \leq n} (w_j - w_i) \prod_{1 \leq i < j \leq n+1} (z_i - z_j)}{\prod_{i=1}^n \prod_{j=1}^{n+1} (w_i - z_j)}. \quad (5.67)$$

Substituting (5.67) into (5.65) gives (5.64). $\square$

Proposition 5.10.2 (General correlator in the (b, c) sector). *Up to an overall normalization,*

$$\left\langle \prod_{i=1}^n b(w_i) \prod_{j=1}^n c(z_j) \right\rangle_{\mathbb{CP}^1} \sim \begin{vmatrix} 2 \log |w_1 - z_1| & \cdots & 2 \log |w_1 - z_n| & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 2 \log |w_n - z_1| & \cdots & 2 \log |w_n - z_n| & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}. \quad (5.68)$$

Proof. Both b and c have one constant zero mode on $\mathbb{CP}^1$. Therefore, one insertion $b(w_r)$ and one insertion $c(z_s)$ must absorb these zero modes. The remaining fields are contracted using Wick's theorem.

Let

$$G_{ij} := 2 \log |w_i - z_j|$$

and denote by $G^{(r,s)}$ the matrix obtained from $G = (G_{ij})_{1 \leq i, j \leq n}$ by deleting its r -th

row and s -th column. Summing over all possible choices of zero-mode insertions gives

$$\left\langle \prod_{i=1}^n b(w_i) \prod_{j=1}^n c(z_j) \right\rangle_{\mathbb{CP}^1} \sim - \sum_{r,s=1}^n (-1)^{r+s} \det G^{(r,s)}. \quad (5.69)$$

Expanding the determinant in (5.68) along its last row and last column gives precisely

$$- \sum_{r,s=1}^n (-1)^{r+s} \det G^{(r,s)}.$$

This proves the result. $\square$

5.11 Q-closed observables and Witten's descent

Proposition 5.11.1 (Correlators of Q -closed fields). *[5, p. 180]*

Let $\Phi_1, \dots, \Phi_n$ be Q -closed fields in a topological conformal field theory, so that

$$Q\Phi_i = 0, \quad i = 1, \dots, n.$$

1. *The correlator*

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle_{\mathbb{CP}^1} \quad (5.70)$$

is constant on the configuration space $C_n(\mathbb{CP}^1)$.

2. *If one of the fields is Q -exact, then the correlator vanishes. More precisely, for any field Ψ ,*

$$\langle Q\Psi(z_1) \Phi_2(z_2) \cdots \Phi_n(z_n) \rangle_{\mathbb{CP}^1} = 0. \quad (5.71)$$

Proof. Since translations are generated by L_{-1} , we have

$$\partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \langle \Phi_1(z_1) \cdots L_{-1} \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle. \quad (5.72)$$

Since $T = Q(G)$, one has

$$L_{-1} = [Q, G_{-1}].$$

Using $Q\Phi_i = 0$, it follows that

$$L_{-1} \Phi_i = Q(G_{-1} \Phi_i).$$

Therefore,

$$\partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \frac{1}{2\pi i} \oint_{\gamma_{z_i}} \langle J(w) \Phi_1(z_1) \cdots G_{-1} \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle dw. \quad (5.73)$$

Deforming the contour around the remaining insertions gives

$$\begin{aligned}
& \frac{1}{2\pi i} \oint_{\gamma_{z_i}} \langle J(w) \Phi_1(z_1) \cdots G_{-1} \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle dw \\
&= - \sum_{j \neq i} \langle \Phi_1(z_1) \cdots Q\Phi_j(z_j) \cdots G_{-1} \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle \\
&= 0,
\end{aligned} \tag{5.74}$$

since $Q\Phi_j = 0$ for all j .

$$\partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = 0.$$

The proof of the second property uses a similar argument. $\square$

By Proposition 5.11.1, a non-vanishing correlator of Q -closed fields implies that none of its insertions is Q -exact.

Example 5.11.1 (A nonvanishing correlator on $\mathbb{CP}^1$). Consider

$$\left\langle \left(\delta(a) \partial c \right) (z_1) \left(b \delta(\gamma) \delta(\bar{\gamma}) \right) (z_2) \left(b \delta(\lambda) \right) (z_3) c(z_4) \right\rangle_{\mathbb{CP}^1}. \tag{5.75}$$

All inserted fields are Q -closed:

$$\begin{aligned}
Q(\delta(a) \partial c) &= -\delta'(a) (\partial c)^2 = 0, & Q(b \delta(\gamma) \delta(\bar{\gamma})) &= \lambda \delta(\gamma) \delta(\bar{\gamma}) = 0, \\
Q(b \delta(\lambda)) &= \lambda \delta(\lambda) = 0, & Qc &= 0.
\end{aligned} \tag{5.76}$$

The correlator factorizes as

$$\begin{aligned}
& \left\langle \left(\delta(a) \partial c \right) (z_1) \left(b \delta(\gamma) \delta(\bar{\gamma}) \right) (z_2) \left(b \delta(\lambda) \right) (z_3) c(z_4) \right\rangle_{\mathbb{CP}^1} \\
&= \langle \partial c(z_1) b(z_2) b(z_3) c(z_4) \rangle_{bc, \mathbb{CP}^1} \\
&\quad \times \langle \delta(a(z_1)) \delta(\gamma(z_2)) \delta(\bar{\gamma}(z_2)) \delta(\lambda(z_3)) \rangle_{a, \gamma, \bar{\gamma}, \mathbb{CP}^1}.
\end{aligned} \tag{5.77}$$

Using (5.68), we have

$$\langle \partial c(z_1) b(z_2) b(z_3) c(z_4) \rangle_{\mathbb{CP}^1} = \frac{z_3 - z_2}{(z_1 - z_2)(z_1 - z_3)}. \tag{5.78}$$

Using (5.64),

$$\langle \delta(a(z_1)) \delta(\gamma(z_2)) \delta(\lambda(z_3)) \rangle_{\mathbb{CP}^1} = \frac{(z_1 - z_2)(z_1 - z_3)}{z_3 - z_2}. \tag{5.79}$$

The insertion $\delta(\bar{\gamma}(z_2))$ absorbs the $\bar{\gamma}$ zero mode and contributes only to the overall

normalization. Fixing this normalization, the two factors cancel and give

$$\left\langle \left(\delta(a) \partial c \right) (z_1) \left(b \delta(\gamma) \delta(\bar{\gamma}) \right) (z_2) \left(b \delta(\lambda) \right) (z_3) c(z_4) \right\rangle_{\mathbb{CP}^1} = 1. \quad (5.80)$$

By Proposition 5.11.1, this nonvanishing correlator implies that none of the inserted fields

$$\delta(a) \partial c, \quad b \delta(\gamma) \delta(\bar{\gamma}), \quad b \delta(\lambda), \quad c$$

is Q -exact on $\mathbb{CP}^1$.

Remark. If we replace $\delta(a) \partial c$ in (5.75) by $\delta(\bar{a}) \bar{\partial} c$, the correlator is again equal to 1. Hence, by Proposition 5.11.1, $\delta(\bar{a}) \bar{\partial} c$ is not Q -exact.

Definition 5.11.1 (Witten's descent of observables). [5, Section 6.6.1] [1, Section 2.5]

Let $\Phi^{(0)}$ be a Q -closed observable. Witten's descent associates to $\Phi^{(0)}$ a tower of differential-form-valued fields

$$\Phi^{(0)}, \quad \Phi^{(1)}, \quad \Phi^{(2)},$$

satisfying the descent equations

$$d\Phi^{(p-1)} = Q\Phi^{(p)}, \quad p = 0, 1, 2, \quad (5.81)$$

with the convention $\Phi^{(-1)} = 0$. Thus,

$$Q\Phi^{(0)} = 0, \quad Q\Phi^{(1)} = d\Phi^{(0)}, \quad Q\Phi^{(2)} = d\Phi^{(1)}. \quad (5.82)$$

Since Σ has real dimension two, the descent tower stops at form degree two.

The descendants define extended observables. Since $\gamma \subset \Sigma$ is a closed one-cycle, then

$$\oint_{\gamma} \Phi^{(1)}$$

is Q -closed. Similarly, if Σ is closed, then

$$\int_{\Sigma} \Phi^{(2)}$$

is also Q -closed. A deformation of the two-dimensional Abelian theory therefore takes the form

$$S \longmapsto S + g \int_{\Sigma} \mathcal{O}^{(2)}, \quad (5.83)$$

where g is a deformation parameter.

To construct an explicit solution of the descent equations, we define

$$\Gamma = -dz G_{-1} - d\bar{z} \bar{G}_{-1} \quad (5.84)$$

The first descendant is then

$$\Phi^{(1)} = \Gamma \Phi^{(0)} = -dz G_{-1} \Phi^{(0)} - d\bar{z} \bar{G}_{-1} \Phi^{(0)} \quad (5.85)$$

The second descendant is

$$\Phi^{(2)} = \frac{1}{2} \Gamma^2 \Phi^{(0)} = -dz \wedge d\bar{z} G_{-1} \bar{G}_{-1} \Phi^{(0)} \quad (5.86)$$

The operators G_{-1} and $\bar{G}_{-1}$ act as graded derivations on the algebra $\mathbb{F}_z^\delta$. Their action on the elementary fields is given by

$$\begin{array}{c|cccccc} & c & \gamma & \bar{\gamma} & b & a & \bar{a} \\ \hline G_{-1} & -a & \partial b & 0 & 0 & 0 & 0 \\ \hline \bar{G}_{-1} & -\bar{a} & 0 & \bar{\partial} b & 0 & 0 & 0 \end{array} \quad (5.87)$$

Their action on general composite fields is then determined by the graded Leibniz rule.

We define the total descendant of $\Phi^{(0)}$ by

$$\tilde{\Phi} = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} \quad (5.88)$$

Example 5.11.2 (Total descendants of delta observables). The total descendants of c and B are

$$\tilde{c} = c + A, \quad \tilde{B} = B - \star db.$$

we also have,

$$\begin{aligned} \widetilde{\delta(B)} &= \delta(B) + i dz \partial b \delta'(B) - i d\bar{z} \bar{\partial} b \delta'(B) \\ &\quad - dz \wedge d\bar{z} \partial b \bar{\partial} b \delta''(B). \end{aligned} \quad (5.89)$$

Similarly, modulo the equations of motion, the total descendant of $\delta(a)\partial c$ is

$$\widetilde{\delta(a)\partial c} \underset{\text{EOM}}{\simeq} \delta(a) \partial c + dz \delta(a) \partial a \quad (5.90)$$

Proposition 5.11.2. [5, section 6.7] Let $\Phi_1, \dots, \Phi_n$ be Q -closed observables. The

correlator of their total descendants,

$$G_n = \left\langle \tilde{\Phi}_1(z_1) \cdots \tilde{\Phi}_n(z_n) \right\rangle, \quad (5.91)$$

defines a closed differential form on the moduli space $\mathcal{M}_{0,n}$. Its total degree is equal to the total ghost number of the product $\Phi_1 \cdots \Phi_n$.

Example 5.11.3. Consider the four-point correlator

$$G_4 = \left\langle \tilde{c}(z_1) \tilde{B}(z_2) b\delta(\gamma)\delta(\bar{\gamma})(z_3) \tilde{c}(z_4) \right\rangle_{\mathbb{CP}^1}.$$

The total ghost number of

$$c(z_1) B(z_2) b\delta(\gamma)\delta(\bar{\gamma})(z_3) c(z_4)$$

is +1. Therefore, G_4 is a one-form on $\mathcal{M}_{0,4}$. Explicitly, we obtain

$$G_4 = 2 d \arg(\xi) \quad (5.92)$$

where $\xi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$, is the cross-ratio of the four marked points z_1, z_2, z_3, z_4 .

Example 5.11.4. Let

$$X = \delta(a) \partial c, \quad \bar{X} = \delta(\bar{a}) \bar{\partial} c, \quad \Theta = b \delta(\gamma) \delta(\bar{\gamma}).$$

Consider the correlator

$$\mathcal{F}(z_1, z_2, z_3, z_4) = \left\langle \widetilde{(X\bar{X})}(z_1) \tilde{\Theta}(z_2) \tilde{\Theta}(z_3) \tilde{c}(z_4) \right\rangle_{\mathbb{CP}^1}.$$

As before, the total ghost number is +1. Hence, $\mathcal{F}$ is a one-form on $\mathcal{M}_{0,4}$. We obtain

$$\mathcal{F}(z_1, z_2, z_3, z_4) = d|\xi| \quad (5.93)$$

Proposition 5.11.3. [1, section 2.6] The space of local 0-observables $\mathcal{O}_z^\delta$ carries the structure of a supercommutative BV algebra. Let $\Phi_1 = \Phi_1^{(0)}$ and $\Phi_2 = \Phi_2^{(0)}$ be two Q -closed observables.

The supercommutative product is induced by the OPE:

$$\Phi_1(w)\Phi_2(z) \sim (\Phi_1 \cdot \Phi_2)(z) \quad \text{modulo } Q\text{-exact terms}, \quad (5.94)$$

where $\Phi_1 \cdot \Phi_2$ is again a Q -cocycle. Thus, in Q -cohomology, the OPE is constant and induces a supercommutative product.

The Poisson bracket of degree -1 is defined by

$$(\Phi_1, \Phi_2)(z) := \frac{1}{2\pi i} \oint_{\gamma_z} \Phi_1^{(1)}(w) \Phi_2^{(0)}(z), \quad (5.95)$$

where γ_z is a contour around z , and

$$\Phi_1^{(1)} = \Gamma \Phi_1^{(0)}$$

is the first descendant of Φ_1 .

The BV Laplacian, also of degree -1 , is defined by

$$\Delta := G_0 - \overline{G}_0 \quad (5.96)$$

For the two-dimensional Abelian BF theory, the only nonvanishing brackets between the generators of $\mathcal{O}_z^\delta$ are

$$\begin{aligned} (c, B) &= -1, & (c, \delta^{(k)}(B)) &= -\delta^{(k+1)}(B), \\ (b, c) &= 1, & (c, \tilde{\mathcal{O}}) &= \frac{i}{2} b \delta'(\gamma) \delta(\bar{\gamma}) - \frac{i}{2} b \delta(\gamma) \delta'(\bar{\gamma}), \end{aligned} \quad (5.97)$$

where

$$\tilde{\mathcal{O}} := b \delta(\gamma) \delta(\bar{\gamma}).$$

Moreover, the BV Laplacian Δ vanishes on all the generators of $\mathcal{O}_z^\delta$.

One of the goals of this thesis is to construct a family of correlators of total descendants of Q -cocycles that are independent of angular variables and extend as d -closed forms to the Deligne–Mumford compactification of $\mathcal{M}_{0,n}$. The only non-trivial correlator we have obtained so far is:

$$\left\langle (\delta(a) \partial c)(z_1) (b \delta(\gamma) \delta(\bar{\gamma}))(z_2) (b \delta(\lambda))(z_3) c(z_4) \right\rangle_{\mathbb{CP}^1} \sim 1 \quad (5.98)$$

Its degree is equal to its total ghost number, which is zero. Therefore, it defines a d -closed 0-form on

$$\mathcal{M}_{0,4} \cong \mathbb{CP}^1 \setminus \{0, 1, \infty\}.$$

Since the correlator is constant, it extends smoothly to the Deligne–Mumford compactification

$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1.$$

The existence of analogous correlators of higher degree remains an open question.

Chapter 6

Two-Dimensional Abelian BF Theory with a Superpotential

6.1 N -Component Abelian BF Theory

We consider the N -component Abelian BF theory by replacing the BRST space of fields $\mathcal{F}_{\text{BRST}}$ in (5.8) by

$$\mathcal{F}_N = \mathcal{F}_{\text{BRST}} \otimes \mathbb{R}^N.$$

The action is

$$S_N = \int_{\Sigma} B_i dA^i + \lambda_i d \star A^i + b_i d \star dc^i. \quad (6.1)$$

The elementary correlators are

$$\langle a^i(u) \gamma_j(z) \rangle_0 = \frac{\delta_j^i}{u - z}, \quad \langle \bar{a}^i(\bar{u}) \bar{\gamma}_j(\bar{z}) \rangle_0 = \frac{\delta_j^i}{\bar{u} - \bar{z}}, \quad \langle c^i(u) b_j(z) \rangle_0 = 2\delta_j^i \log \frac{|u - z|}{R}. \quad (6.2)$$

We deform the N -component Abelian theory by introducing a superpotential in the ghost fields c^i , inspired by the approach of Losev, Mnev, and Youmans for the non-Abelian theory [1, 15].

6.2 Superpotential Deformation

Let

$$\mathcal{O}^{(0)} = W(c),$$

where W is a polynomial containing only even monomials in the fields c^i :

$$W(c) = \sum_{p \geq 1} W_{2p}(c), \quad W_{2p}(c) = \frac{1}{(2p)!} \mu_{i_1 \dots i_{2p}} c^{i_1} \dots c^{i_{2p}}. \quad (6.3)$$

By Witten's descent, the first descendant is

$$\mathcal{O}^{(1)} = A^i \frac{\partial W}{\partial c^i}. \quad (6.4)$$

The second descendant is

$$\mathcal{O}^{(2)} = -\frac{1}{2} A^j \wedge A^k \frac{\partial^2 W(c)}{\partial c^j \partial c^k}. \quad (6.5)$$

We deform the action by the integrated second descendant:

$$S_g = S_0 + g \int_{\Sigma} \mathcal{O}^{(2)} = S_0 - \frac{g}{2} \int_{\Sigma} A^i \wedge A^k \frac{\partial^2 W(c)}{\partial c^i \partial c^k}. \quad (6.6)$$

6.3 Equations of Motion

The equations of motion of the deformed theory are

$$\begin{aligned} \bar{\partial} a^k &= 0, & \partial \bar{a}^k &= 0, & \partial \bar{\partial} c^k &= 0, \\ 2i \bar{\partial} \gamma_k + g \bar{a}^j \frac{\partial^2 W(c)}{\partial c^k \partial c^j} &= 0, & 2i \partial \bar{\gamma}_k + g a^i \frac{\partial^2 W(c)}{\partial c^i \partial c^k} &= 0, \\ 2i \partial \bar{\partial} b_k + g a^i \bar{a}^j \frac{\partial^3 W(c)}{\partial c^k \partial c^i \partial c^j} &= 0. \end{aligned} \quad (6.7)$$

6.4 BRST Operator and BRST Current

6.4.1 BRST Operator

We write the deformed BRST differential and action as

$$Q_g = Q_0 + gQ_1, \quad S_g = S_0 + gS_1, \quad (6.8)$$

where Q_0 is the BRST differential of the free Abelian theory and,

$$S_1 = -\frac{1}{2} \int_{\Sigma} A^i \wedge A^k \frac{\partial^2 W(c)}{\partial c^i \partial c^k}. \quad (6.9)$$

BRST invariance at first order in g requires

$$Q_0 S_1 + Q_1 S_0 = 0. \quad (6.10)$$

Using $Q_0 A^i = dc^i$, we obtain

$$Q_0 S_1 = - \int_{\Sigma} dc^i \wedge A^k \frac{\partial^2 W(c)}{\partial c^i \partial c^k}.$$

Since

$$A^k \frac{\partial^2 W(c)}{\partial c^i \partial c^k} dc^i = A^k d \left(\frac{\partial W}{\partial c^k} \right),$$

integration by parts gives

$$Q_0 S_1 = \int_{\Sigma} dA^k \frac{\partial W}{\partial c^k}.$$

This contribution can be cancelled by varying B_k in the free action. We therefore choose

$$Q_1 B_k = - \frac{\partial W}{\partial c^k}. \quad (6.11)$$

The remaining BRST transformations are unchanged. Hence

$$\begin{aligned} Q_g A^i &= dc^i, & Q_g c^i &= 0, & Q_g B_i &= -g \frac{\partial W}{\partial c^i}, \\ Q_g b_i &= \lambda_i, & Q_g \lambda_i &= 0. \end{aligned} \quad (6.12)$$

It follows immediately from (6.12) that $Q_g^2 = 0$ off shell.

6.4.2 BRST Current

The deformed BRST current is

$$J_{\text{BRST}}^{(g)} = J_{\text{BRST}}^{(0)} + g A^i \frac{\partial W}{\partial c^i}. \quad (6.13)$$

where $J_{\text{BRST}}^{(0)}$ is the conserved BRST current of the free theory.

In complex coordinates, we write $J_{\text{BRST}}^{(g)} = dz J^{(g)} + d\bar{z} \bar{J}^{(g)}$, where

$$J^{(g)} = -2i \gamma_i \partial c^i + g a^i \frac{\partial W}{\partial c^i}, \quad \bar{J}^{(g)} = 2i \bar{\gamma}_i \bar{\partial} c^i + g \bar{a}^i \frac{\partial W}{\partial c^i}. \quad (6.14)$$

In the deformed theory, the chiral conservation laws (5.21) do no longer hold separately for $J^{(g)}$ and $\bar{J}^{(g)}$.

Proposition 6.4.1. *The deformed BRST current admits the decomposition*

$$J_{\text{BRST}}^{(g)} = J_L^{(g)} + J_R^{(g)}, \quad (6.15)$$

where

$$J_L^{(g)} = -2i dz \gamma_i \partial c^i + g d\bar{z} \bar{a}^i \frac{\partial W}{\partial c^i}, \quad J_R^{(g)} = g dz a^i \frac{\partial W}{\partial c^i} + 2i d\bar{z} \bar{\gamma}_i \bar{\partial} c^i. \quad (6.16)$$

Both currents are separately d -closed on shell:

$$dJ_L^{(g)} \underset{\text{EOM}}{\sim} 0, \quad dJ_R^{(g)} \underset{\text{EOM}}{\sim} 0.$$

Proof. Using the definition of $J_L^{(g)}$ in (6.16), we obtain

$$\begin{aligned} dJ_L^{(g)} &= dz \wedge d\bar{z} \left[2i \bar{\partial} (\gamma_i \partial c^i) + g \partial \left(\bar{a}^i \frac{\partial W}{\partial c^i} \right) \right] \\ &= dz \wedge d\bar{z} \left[2i \bar{\partial} \gamma_i \partial c^i + 2i \gamma_i \bar{\partial} \partial c^i + g (\partial \bar{a}^i) \frac{\partial W}{\partial c^i} + g \bar{a}^i \frac{\partial^2 W}{\partial c^j \partial c^i} \partial c^j \right]. \end{aligned} \quad (6.17)$$

Using the equations of motion (6.7),

$$\bar{\partial} \partial c^i = 0, \quad \partial \bar{a}^i = 0,$$

we find

$$dJ_L^{(g)} = dz \wedge d\bar{z} \left(2i \bar{\partial} \gamma_j + g \bar{a}^i \frac{\partial^2 W}{\partial c^j \partial c^i} \right) \partial c^j.$$

The expression in parentheses vanishes by the equation of motion for γ_j , and therefore

$$dJ_L^{(g)} = 0.$$

Since the full BRST current is conserved and $J_{\text{BRST}}^{(g)} = J_L^{(g)} + J_R^{(g)}$, it follows that

$$dJ_R^{(g)} = 0.$$

□

6.5 Stress-Energy Tensor

The interaction

$$g \int_{\Sigma} \mathcal{O}^{(2)}$$

is independent of the metric. Therefore, the metric dependence of S_g is the same as in the free Abelian theory and comes only from the Hodge star in the gauge-fixing term (5.11).

The deformation leaves $G_{\text{tot}} = (dz)^2 a^i \partial b_i + (d\bar{z})^2 \bar{a}^i \bar{\partial} b_i$ unchanged. Since

$$Q_g a^i = -\partial c^i, \quad Q_g \bar{a}^i = -\bar{\partial} c^i, \quad Q_g b_i = \lambda_i,$$

we obtain

$$\begin{aligned} Q_g G_{\text{tot}} &= (dz)^2 T + (d\bar{z})^2 \bar{T}, \\ T &= a^i \partial \lambda_i + \partial b_i \partial c^i, \quad \bar{T} = \bar{a}^i \bar{\partial} \lambda_i + \bar{\partial} b_i \bar{\partial} c^i. \end{aligned} \tag{6.18}$$

Here T and $\bar{T}$ denote the holomorphic and antiholomorphic components of the stress-energy tensor, respectively. By (6.7),

$$\bar{\partial} T \underset{\text{EOM}}{\sim} 0, \quad \partial \bar{T} \underset{\text{EOM}}{\sim} 0. \tag{6.19}$$

6.6 Examples of Correlators

The interaction term is

$$S_{\text{int}} = -g \int a^i \bar{a}^j \frac{\partial^2 W(c)}{\partial c^i \partial c^j}.$$

Since

$$\frac{\partial^2 W(c)}{\partial c^i \partial c^j} = \sum_{p \geq 1} \frac{1}{(2p-2)!} \mu_{j i i_3 \dots i_{2p}} c^{i_3} \dots c^{i_{2p}},$$

we obtain

$$S_{\text{int}} = -g \sum_{p \geq 1} \int \frac{1}{(2p-2)!} \mu_{j i i_3 \dots i_{2p}} a^i \bar{a}^j c^{i_3} \dots c^{i_{2p}}.$$

In the diagrams below, thin solid lines represent a -legs contracted with γ -insertions, thick solid lines represent $\bar{a}$ -legs contracted with $\bar{\gamma}$ -insertions, and dashed lines represent c -legs contracted with b -insertions.

Two-Point Correlator

We first consider

$$\begin{aligned} \langle \gamma_i(z) \bar{\gamma}_j(w) \rangle_g &= \frac{g}{4\pi} \sum_{p \geq 1} \frac{1}{(2p-2)!} \int \mu_{j i i_3 \dots i_{2p}} \langle a^i(u) \bar{a}^j(u) c^{i_3}(u) \dots c^{i_{2p}}(u) \gamma_i(z) \bar{\gamma}_j(w) \rangle_0 \\ &= \frac{g}{4\pi} \mu_{ji} \int \frac{d^2 u}{(u-z)(\bar{u}-\bar{w})}. \end{aligned} \tag{6.20}$$

The integral over $\mathbb{C}$ is infrared divergent. We regularize it, as in the prescription used in [15, Appendix A], by replacing $\mathbb{C}$ with the disk $D_R = \{u \in \mathbb{C} \mid |u| \leq R\}$.

This gives

$$\begin{aligned}\langle \gamma_i(z) \bar{\gamma}_j(w) \rangle_g &= \frac{g}{4\pi} \mu_{ji} \log \frac{R^2 - z\bar{w}}{|z - w|^2} \\ &= \frac{g}{4} \mu_{ij} \log \frac{|z - w|^2}{R^2} + \mathcal{O}(R^{-2}), \quad R \rightarrow \infty.\end{aligned}\quad (6.21)$$

The corresponding connected diagram is

$$\gamma_i(z) \longrightarrow W_2(u) \longleftarrow \bar{\gamma}_j(w),$$

with

$$W_2(u) = \mu_{ji} a^i(u) \bar{a}^j(u).$$

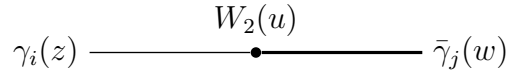


Figure 6.1: Feynman diagram for $\langle \gamma_i(z) \bar{\gamma}_j(w) \rangle_g$.

Four-Point Correlator

We now consider

$$\begin{aligned}\langle \gamma_m(z) \bar{\gamma}_n(w) b_r(y) b_s(x) \rangle_g &= \frac{g}{8\pi} \int \mu_{ji\alpha\beta} \langle a^i(u) \bar{a}^j(u) c^\alpha(u) c^\beta(u) \gamma_m(z) \bar{\gamma}_n(w) b_r(y) b_s(x) \rangle_0 \\ &= \frac{g}{8\pi} \int \mu_{ji\alpha\beta} \frac{\delta_m^i}{u - z} \frac{\delta_n^j}{\bar{u} - \bar{w}} \langle c^\alpha(u) c^\beta(u) b_r(y) b_s(x) \rangle_0 \\ &= \frac{g}{8\pi} \int \mu_{ji\alpha\beta} \frac{\delta_m^i \delta_n^j}{(u - z)(\bar{u} - \bar{w})} \left(-\langle c^\alpha(u) b_r(y) \rangle_0 \langle c^\beta(u) b_s(x) \rangle_0 \right. \\ &\quad \left. + \langle c^\alpha(u) b_s(x) \rangle_0 \langle c^\beta(u) b_r(y) \rangle_0 \right) d^2u \\ &= \frac{2g}{\pi} \mu_{nmrs} \int \frac{\log \left| \frac{u - x}{R} \right| \log \left| \frac{u - y}{R} \right|}{(u - z)(\bar{u} - \bar{w})} d^2u.\end{aligned}\quad (6.22)$$

This correlator also has an infrared divergence.

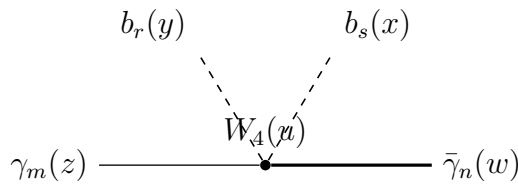


Figure 6.2: Feynman diagram for $\langle \gamma_m(z) \bar{\gamma}_n(w) b_r(y) b_s(x) \rangle_g$.

General $2p$ -Point Correlator

Consider

$$\left\langle \gamma_m(z) \bar{\gamma}_n(w) b_{j_3}(x_3) \cdots b_{j_{2p}}(x_{2p}) \right\rangle_g.$$

Then

$$\begin{aligned} & \left\langle \gamma_m(z) \bar{\gamma}_n(w) b_{j_3}(x_3) \cdots b_{j_{2p}}(x_{2p}) \right\rangle_g \\ &= \frac{g}{4\pi(2p-2)!} \int \mu_{j i i_3 \cdots i_{2p}} \frac{\delta_m^i \delta_n^j}{(u-z)(\bar{u}-\bar{w})} \left\langle c^{i_3}(u) \cdots c^{i_{2p}}(u) b_{j_3}(x_3) \cdots b_{j_{2p}}(x_{2p}) \right\rangle_0 \\ &= \frac{g}{4\pi(2p-2)!} \int \frac{\mu_{j i i_3 \cdots i_{2p}}}{(u-z)(\bar{u}-\bar{w})} \sum_{\sigma \in S_{2p-2}} (-1)^{|\sigma|} \prod_{k=3}^{2p} \left\langle c^{i_k}(u) b_{j_{\sigma(k)}}(x_{\sigma(k)}) \right\rangle_0 d^2 u \\ &= \frac{g}{4\pi} \mu_{nmj_3 \cdots j_{2p}} \int \frac{\prod_{k=3}^{2p} 2 \log \left| \frac{u-x_k}{R} \right|}{(u-z)(\bar{u}-\bar{w})} d^2 u. \end{aligned} \quad (6.23)$$

For large $|u|$,

$$|u| \gg |x_k|, \quad |u| \gg |z|, \quad |u| \gg |w|,$$

we have

$$\frac{\prod_{k=3}^{2p} \log \left| \frac{u-x_k}{R} \right|^2}{(u-z)(\bar{u}-\bar{w})} \sim \frac{\log^{2p-2} |u|^2}{|u|^2}.$$

In polar coordinates, $u = r e^{i\theta}$ and $d^2 u = r dr d\theta$,

$$\begin{aligned} \int \frac{\log^{2p-2}(r^2)}{r^2} r dr d\theta &\sim 2\pi \int \frac{\log^{2p-2}(r/R)}{r} dr \\ &\sim \frac{2\pi}{2p-1} \log^{2p-1} \frac{r}{R} \end{aligned} \quad (6.24)$$

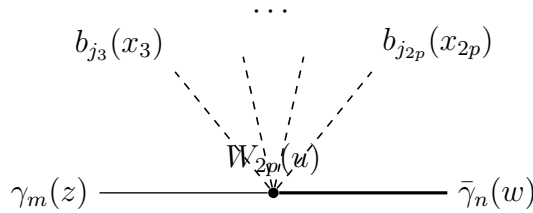


Figure 6.3: Feynman diagram contributing to $\left\langle \gamma_m(z) \bar{\gamma}_n(w) b_{j_3}(x_3) \cdots b_{j_{2p}}(x_{2p}) \right\rangle_g$.

Correlator on the Sphere $\mathbb{CP}^1$

We now investigate this infrared divergence on the sphere. Consider

$$\begin{aligned}
& \langle \delta(\gamma_i)(z_1) \gamma_i(z) \delta(\bar{\gamma}_j)(z_2) \bar{\gamma}_j(w) \rangle_g \\
&= \frac{g}{4\pi} \int \mu_{kn} \langle a^n(u) \delta(\gamma_i)(z_1) \gamma_i(z) \rangle_0 \langle \bar{a}^k(u) \delta(\bar{\gamma}_j)(z_2) \bar{\gamma}_j(w) \rangle_0 d^2u \\
&= \frac{g}{4\pi} \int \mu_{ji} \left(\frac{1}{u-z} - \frac{1}{u-z_1} \right) \left(\frac{1}{\bar{u}-\bar{w}} - \frac{1}{\bar{u}-\bar{z}_2} \right) d^2u \\
&= \frac{g\mu_{ji}}{4\pi} \int \left[\frac{1}{(u-z)(\bar{u}-\bar{w})} - \frac{1}{(u-z)(\bar{u}-\bar{z}_2)} \right. \\
&\quad \left. - \frac{1}{(u-z_1)(\bar{u}-\bar{w})} + \frac{1}{(u-z_1)(\bar{u}-\bar{z}_2)} \right] d^2u. \quad (6.25)
\end{aligned}$$

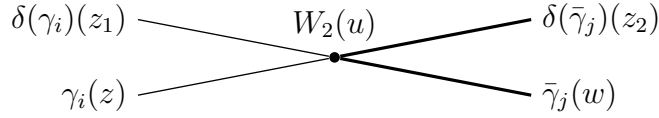


Figure 6.4: Feynman diagram contributing to $\langle \delta(\gamma_i)(z_1) \gamma_i(z) \delta(\bar{\gamma}_j)(z_2) \bar{\gamma}_j(w) \rangle_g$.

A new Q -closed Observable

For the W -deformation, a new Q -closed observable is $W_i(c) \delta(B_i)$.

Consider

$$\begin{aligned}
& \left\langle \gamma_i(z_1) \bar{\gamma}_j(z_2) \prod_{r=1}^N b_r(x_r) \delta(\lambda_r)(x_r) \prod_{r=1}^N W_r(c)(y_r) \delta(B_r)(y_r) \right\rangle_g \\
& \sim \int \left\langle a^i(u) \gamma_i(z_1) \delta(\lambda_i)(x_i) \delta(B_i)(y_i) \right\rangle_0 \left\langle \bar{a}^j(u) \bar{\gamma}_j(z_2) \delta(\lambda_j)(x_j) \delta(B_j)(y_j) \right\rangle_0 d^2u. \quad (6.26)
\end{aligned}$$

We have

$$\left\langle a^i(u) \gamma_i(z_1) \delta(\lambda_i)(x_i) \delta(B_i)(y_i) \right\rangle_0 \sim \frac{2}{u-z_1} - \frac{1}{u-y_i} - \frac{1}{u-x_i},$$

and similarly,

$$\left\langle \bar{a}^j(u) \bar{\gamma}_j(z_2) \delta(\lambda_j)(x_j) \delta(B_j)(y_j) \right\rangle_0 \sim \frac{2}{\bar{u}-\bar{z}_2} - \frac{1}{\bar{u}-\bar{y}_j} - \frac{1}{\bar{u}-\bar{x}_j}.$$

Therefore,

$$\begin{aligned} & \left\langle \gamma_i(z_1) \bar{\gamma}_j(z_2) \prod_{r=1}^N b_r(x_r) \delta(\lambda_r)(x_r) \prod_{r=1}^N W_r(c)(y_r) \delta(B_r)(y_r) \right\rangle_g \\ & \sim \int \left(\frac{2}{u - z_1} - \frac{1}{u - y_i} - \frac{1}{u - x_i} \right) \left(\frac{2}{\bar{u} - \bar{z}_2} - \frac{1}{\bar{u} - \bar{y}_j} - \frac{1}{\bar{u} - \bar{x}_j} \right) d^2 u. \end{aligned} \quad (6.27)$$

This integral is ultraviolet- and infrared-finite.

For $x_i = y_i$ and $x_j = y_j$, it becomes

$$\int \left(\frac{1}{u - z_1} - \frac{1}{u - x_i} \right) \left(\frac{1}{\bar{u} - \bar{z}_2} - \frac{1}{\bar{u} - \bar{x}_j} \right) d^2 u.$$

Sending both x_i and x_j to infinity reproduces the two-point correlator computed previously. The infrared divergence can therefore be understood as arising from the zero-mode fixing prescription used when passing from $\mathbb{CP}^1$ to $\mathbb{C}$: If the soaking insertions are placed at the same point, their operator product with the interaction becomes singular. Keeping their insertion points separated therefore provides a natural point-splitting regularization.

For instance, taking

$$x_i = \infty, \quad x_j \in \mathbb{CP}^1 \setminus \{\infty\},$$

the corresponding propagator is

$$\begin{aligned} \langle \gamma_i(z_1) \bar{\gamma}_j(z_2) \rangle_{x_j} &= \int \frac{1}{u - z_1} \left(\frac{1}{\bar{u} - \bar{z}_2} - \frac{1}{\bar{u} - \bar{x}_j} \right) d^2 u \\ &= 2 \log \left| \frac{z_1 - z_2}{z_1 - x_j} \right|. \end{aligned} \quad (6.28)$$

6.7 Feynman Rules

Recall from (6.3) that

$$W(c) = \sum_{p \geq 1} W_{2p}(c),$$

where W_{2p} is homogeneous of degree $2p$ in the fields c^i .

The interaction associated with W_{2p} is

$$V_{2p}(u) = -\frac{g}{(2p-2)!} \mu_{j i i_3 \dots i_{2p}} a^i(u) \bar{a}^j(u) c^{i_3}(u) \dots c^{i_{2p}}(u). \quad (6.29)$$

Each interaction vertex contains one a -field, one $\bar{a}$ -field, and $2p-2$ c -fields. Since

the only non-vanishing free contractions are

$$a \longleftrightarrow \gamma, \quad \bar{a} \longleftrightarrow \bar{\gamma}, \quad c \longleftrightarrow b,$$

a V_{2p} -vertex is connected to one γ -leg, one $\bar{\gamma}$ -leg, and $2p - 2$ b -legs.

Two interaction vertices cannot contract with each other, since

$$V_{2p}(u)V_{2q}(v) \sim \text{regular}.$$

Hence no ultraviolet divergence arises from collisions of integrated vertices. Infrared divergences may still arise in correlators containing γ , $\bar{\gamma}$, or b legs.

For correlators of elementary fields, every interaction vertex forms a one-vertex star diagram as shown in the previous examples. We can conclude that every non-vanishing Feynman diagram of such correlators is a finite disjoint union of such stars and purely free propagator lines.

6.7.1 Selection Rules

Consider a correlator of the following form:

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle_g, \quad \Phi_i \in \{a, \bar{a}, \gamma, \bar{\gamma}, b, c\}. \quad (6.30)$$

Let N_Φ denote the number of insertions of the field Φ . If a non-vanishing diagram contains V interaction vertices, then the a - γ and $\bar{a}$ - $\bar{\gamma}$ contractions imply

$$N_\gamma - N_a = N_{\bar{\gamma}} - N_{\bar{a}} = V \geq 0. \quad (6.31)$$

Suppose that the r -th interaction vertex comes from the homogeneous component W_{2p_r} . Since this vertex contains $2p_r - 2$ c -fields, the b - c contractions further require

$$N_b - N_c = \sum_{r=1}^V (2p_r - 2). \quad (6.32)$$

Proposition 6.7.1 (Correlators of elementary fields). *Consider a correlator (6.30) with pairwise distinct insertion points.*

1. *The correlator receives contributions from only finitely many Feynman diagrams. Every diagram is a finite disjoint union of one-vertex star diagrams.*
2. *The corresponding Feynman integrals are ultraviolet finite at separated points.*

Proof. The claim follows immediately from the preceding discussion. $\square$

Proposition 6.7.2. *Consider a correlator (6.30) with pairwise distinct insertion points.*

1. *If every γ -field in the correlator is differentiated, then every contributing diagram is infrared convergent. The same statement holds if every $\bar{\gamma}$ -field is differentiated.*
2. *If the quadratic component is absent, $W_2 = 0$, then differentiating every b -field in the correlator is sufficient to make every contributing diagram infrared convergent.*

Proof. 1. By the power-counting argument of Subsection 6.6, a W_{2p} vertex behaves for large $|u|$ as

$$\frac{\log^{2p-2} |u|}{|u|^2}.$$

Differentiating every γ -field replaces one factor $1/(u - z)$ by $1/(u - z)^2$, so the integrand behaves as

$$\frac{\log^{2p-2} |u|}{|u|^3},$$

which is infrared convergent. The same argument applies when every $\bar{\gamma}$ -field is differentiated.

2. If $W_2 = 0$, only vertices W_{2p} with $p \geq 2$ contribute. Differentiating every b -field replaces each logarithmic factor by a factor of order $|u|^{-1}$. By the same power-counting argument, a W_{2p} vertex then behaves as

$$\frac{1}{|u|^{2p}},$$

whose radial integral is convergent for $p \geq 2$.

□

Example

Consider the correlator

$$\langle \gamma_{m_1}(z_1) \gamma_{m_2}(z_2) \gamma_{m_3}(z_3) \bar{\gamma}_{n_1}(w_1) \bar{\gamma}_{n_2}(w_2) \bar{\gamma}_{n_3}(w_3) b_{r_1}(y_1) b_{r_2}(y_2) b_{r_3}(y_3) b_{r_4}(y_4) \rangle_g. \quad (6.33)$$

Since

$$N_\gamma = N_{\bar{\gamma}} = 3,$$

the selection rule (6.31) implies that a non-vanishing diagram must contain exactly

three interaction vertices,

$$V_{2p_1}, \quad V_{2p_2}, \quad V_{2p_3}.$$

On the other hand, there are four b -insertions and no external c -fields. Hence (6.32) gives

$$(2p_1 - 2) + (2p_2 - 2) + (2p_3 - 2) = 4,$$

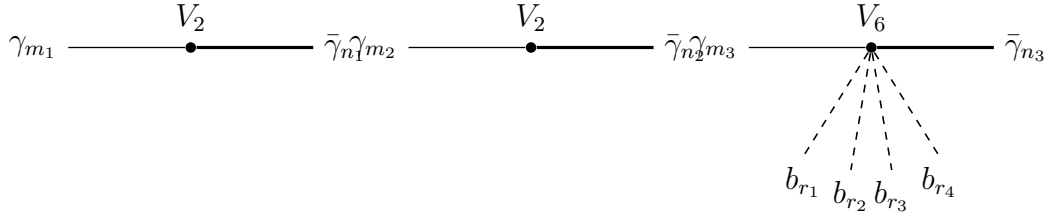
Up to permutations of the three vertices, there are therefore two possibilities:

$$(2p_1, 2p_2, 2p_3) = (2, 2, 6) \quad \text{or} \quad (2, 4, 4). \quad (6.34)$$

The corresponding diagrams are shown below.

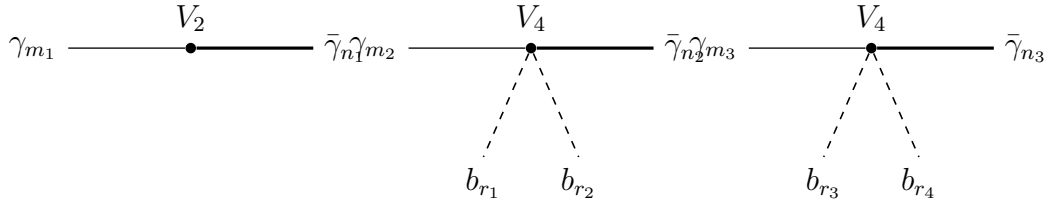
First possibility:

$$V_2 \sqcup V_2 \sqcup V_6.$$



Second possibility:

$$V_2 \sqcup V_4 \sqcup V_4.$$



6.8 Operator Product Expansions

The deformation generates new OPEs between the elementary fields. We denote

$$W_{i_1 \dots i_k}(c) = \frac{\partial^k W(c)}{\partial c^{i_1} \dots \partial c^{i_k}}.$$

The $\gamma\bar{\gamma}$ OPE

At first order in g ,

$$\gamma_i(w)\bar{\gamma}_j(z) \sim \frac{g}{4\pi} \int d^2u \frac{W_{ij}(c(u))}{(u-w)(\bar{u}-\bar{z})}.$$

Expanding around z gives

$$\gamma_i(w)\bar{\gamma}_j(z) \sim -\frac{g}{4} [W_{ij}(c(z)) + (w-z)\partial W_{ij}(c(z))] \log \frac{|w-z|^2}{R^2} + \dots. \quad (6.35)$$

Similarly,

$$\bar{\gamma}_i(w)\gamma_j(z) \sim \frac{g}{4} [W_{ij}(c(z)) + (\bar{w}-\bar{z})\bar{\partial} W_{ij}(c(z))] \log \frac{|w-z|^2}{R^2} + \dots. \quad (6.36)$$

Since $\lambda_i = \gamma_i + \bar{\gamma}_i$, the leading logarithmic terms cancel:

$$\lambda_i(w)\lambda_j(z) \sim \gamma_i(w)\bar{\gamma}_j(z) + \bar{\gamma}_i(w)\gamma_j(z).$$

Hence

$$\lambda_i(w)\lambda_j(z) \sim -\frac{g}{4} \log \frac{|w-z|^2}{R^2} [(w-z)\partial W_{ij}(c(z)) - (\bar{w}-\bar{z})\bar{\partial} W_{ij}(c(z))] + \dots. \quad (6.37)$$

The symbol $\dots$ denotes subleading terms in the separations $w-z$ and $\bar{w}-\bar{z}$, including terms of the form

$$(w-z)^k(\bar{w}-\bar{z})^\ell \log |w-z|^2, \quad k, \ell \geq 0, \quad k+\ell \geq 2,$$

together with terms regular at $w=z$. These terms are continuous at $w=z$ but are not, in general, smooth there. They must therefore be retained when deriving OPEs involving derivatives of the elementary fields, since differentiation can turn them into singular terms.

The γb OPE

We have

$$\begin{aligned} \gamma_i(w)b_j(z) &\sim \frac{g}{4\pi} \int d^2u a^p(u)\bar{a}^k(u)W_{pk}(c(u))\gamma_i(w)b_j(z) \\ &\sim -\frac{g}{4\pi} \int d^2u \frac{\log\left(\frac{|u-z|^2}{R^2}\right)}{u-w} \bar{a}^k(u)W_{jik}(c(u)). \end{aligned} \quad (6.38)$$

Using

$$\int d^2u \frac{\log\left(\frac{|u-z|^2}{R^2}\right)}{u-w} = -\pi(\bar{w}-\bar{z}) \left[\log \frac{|w-z|^2}{R^2} - 1 \right] + \text{regular},$$

we obtain

$$\gamma_i(w)b_j(z) \sim \frac{g}{4}(\bar{w}-\bar{z}) \log \frac{|w-z|^2}{R^2} \bar{a}^k W_{jik}(c(z)) + \dots \quad (6.39)$$

Similarly,

$$\bar{\gamma}_i(w)b_j(z) \sim \frac{g}{4}(w-z) \log \frac{|w-z|^2}{R^2} a^k W_{jki}(c(z)) + \dots \quad (6.40)$$

Here the omitted logarithmic terms have total order at least two in $w-z$ and $\bar{w}-\bar{z}$.

The bb OPE

The double contraction of the two b -fields with the interaction vertex gives

$$b_i(w)b_j(z) \sim \frac{g}{4} a^p(z) \bar{a}^q(z) W_{jipq}(c(z)) |w-z|^2 \log \frac{|w-z|^2}{R^2} + \dots \quad (6.41)$$

Here the omitted logarithmic terms have total order at least three in $w-z$ and $\bar{w}-\bar{z}$.

6.9 Quantum Equations of Motion

We now verify that the classical equations of motion (6.7) continue to hold at the quantum level, namely inside correlators away from the insertion points.

More precisely, if E_Φ denotes the equation of motion associated with a field Φ , we want to show that

$$\langle E_\Phi(z) \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle_g = 0, \quad z \neq z_1, \dots, z_n, \quad (6.42)$$

Proposition 6.9.1. *The classical equations of motion (6.7) hold at the quantum level inside correlators, away from the insertion points.*

Proof. We first consider the equation of motion for γ_k . Expanding perturbatively,

$$\langle 2i \bar{\partial} \gamma_k(z) \Phi_1 \cdots \Phi_n \rangle_g = \sum_{N \geq 1} \frac{g^N}{(4\pi)^N N!} \int_{u_1} \cdots \int_{u_N} \left\langle 2i \bar{\partial} \gamma_k(z) \prod_{r=1}^N \mathcal{O}^{(2)}(u_r) \Phi_1 \cdots \Phi_n \right\rangle_0.$$

At order N , there are N possible interaction vertices with which $\bar{\partial}\gamma_k(z)$ can contract. For one such vertex, we obtain

$$\begin{aligned}
& 2i \bar{\partial}\gamma_k(z) \int_{u_r} \mathcal{O}^{(2)}(u_r) \\
& \longrightarrow -2i \int \bar{\partial}_z \left(\frac{1}{u_r - z} \right) \bar{a}^j(u_r) \frac{\partial^2 W}{\partial c^k \partial c^j}(u_r) \\
& = -2i\pi \int \delta^{(2)}(z - u_r) \bar{a}^j(u_r) \frac{\partial^2 W}{\partial c^k \partial c^j}(u_r) du_r d\bar{u}_r \\
& = -4\pi \bar{a}^j(z) \frac{\partial^2 W}{\partial c^k \partial c^j}(z).
\end{aligned} \tag{6.43}$$

Therefore

$$\begin{aligned}
& \left\langle 2i \bar{\partial}\gamma_k(z) \Phi_1 \cdots \Phi_n \right\rangle_g \\
& = \sum_{N \geq 1} \frac{g^N N (-4\pi)}{(4\pi)^N N!} \left\langle \bar{a}^j \frac{\partial^2 W}{\partial c^k \partial c^j} \prod_{r=1}^{N-1} \mathcal{O}^{(2)}(u_r) \Phi_1 \cdots \Phi_n \right\rangle_0 \\
& = -g \sum_{N \geq 1} \frac{g^{N-1}}{(4\pi)^{N-1} (N-1)!} \left\langle \bar{a}^j \frac{\partial^2 W}{\partial c^k \partial c^j} \prod_{r=1}^{N-1} \mathcal{O}^{(2)}(u_r) \Phi_1 \cdots \Phi_n \right\rangle_0 \\
& = -g \left\langle \bar{a}^j(z) \frac{\partial^2 W}{\partial c^k \partial c^j}(z) \Phi_1 \cdots \Phi_n \right\rangle_g.
\end{aligned} \tag{6.44}$$

Hence

$$\left\langle \left(2i \bar{\partial}\gamma_k + g \bar{a}^j \frac{\partial^2 W}{\partial c^k \partial c^j} \right) (z) \Phi_1 \cdots \Phi_n \right\rangle_g = 0. \tag{6.45}$$

The equation of motion for $\bar{\gamma}_k$ follows analogously.

Equation of motion for b_k .

The same reasoning applies to b_k . We have

$$\begin{aligned}
& \int 2i \partial \bar{\partial} b_k(z) \mathcal{O}^{(2)}(u_r) \\
& \longrightarrow 2i \int a^i(u_r) \bar{a}^j(u_r) \partial \bar{\partial} \log |z - u_r|^2 \frac{\partial^3 W}{\partial c^k \partial c^i \partial c^j}(u_r).
\end{aligned} \tag{6.46}$$

Using

$$\partial \bar{\partial} \log |z - u_r|^2 = \pi \delta^{(2)}(z - u_r),$$

the interaction vertex collapses at $u_r = z$. Repeating the same combinatorial argument as above gives

$$\left\langle \left(2i \partial \bar{\partial} b_k + g a^i \bar{a}^j \frac{\partial^3 W}{\partial c^k \partial c^i \partial c^j} \right) (z) \Phi_1 \cdots \Phi_n \right\rangle_g = 0. \tag{6.47}$$

The remaining equations in (6.7) are linear and follow in the same way. Hence all the classical equations of motion hold inside quantum correlators away from contact terms. $\square$

Corollary 6.9.2. *The holomorphic and antiholomorphic components of G_{tot} and of the stress-energy tensor remain conserved at the quantum level:*

$$\bar{\partial}G = 0, \quad \partial\bar{G} = 0, \quad \bar{\partial}T = 0, \quad \partial\bar{T} = 0, \quad (6.48)$$

inside correlators away from contact terms.

6.10 OPEs of The TCFT Currents

We recall the holomorphic BRST current and the field G ,

$$J^{(g)} = -2i \gamma_i \partial c^i + g a^i \frac{\partial W}{\partial c^i}, \quad G = a^i \partial b_i. \quad (6.49)$$

The holomorphic component of the stress tensor is initially written as

$$T = a^i \partial \lambda_i + \partial b_i \partial c^i, \quad \lambda_i = \gamma_i + \bar{\gamma}_i. \quad (6.50)$$

Using the equation of motion for $\bar{\gamma}_i$, we have

$$a^i \partial \bar{\gamma}_i = 0,$$

and therefore, on shell,

$$T \simeq a^i \partial \gamma_i + \partial b_i \partial c^i. \quad (6.51)$$

In the following OPE computations, we use this on-shell expression for T . The composite fields $J^{(g)}$ and G , together with their antiholomorphic counterparts $\bar{J}^{(g)}$ and $\bar{G}$, do not require any normal ordering.

The stress-energy tensor T , on the other hand, is defined using a point-splitting prescription. The singular contributions arising from the $a^i \partial \gamma_i$ and $\partial b_i \partial c^i$ sectors cancel against each other, so that the final expression for T does not require any additional renormalization.

The $J^{(g)}(w)G(z)$ OPE

We have

$$\begin{aligned}
J^{(g)}(w)G(z) &\sim \left(-2i \gamma_i \partial c^i + g a^i \frac{\partial W}{\partial c^i} \right) (w) (a^j \partial b_j) (z) \\
&\sim \frac{2i}{w-z} (\partial c^j \partial b_j) (z) - \frac{2i}{(w-z)^2} a^i(z) \gamma_i(w) + \frac{2iN}{(w-z)^3} \\
&\quad + \frac{g}{w-z} a^i(w) a^j(z) \frac{\partial^2 W}{\partial c^j \partial c^i} (w) + \text{reg.}
\end{aligned} \tag{6.52}$$

Since

$$a^i a^j \frac{\partial^2 W}{\partial c^j \partial c^i} = 0$$

by antisymmetry of $\partial^2 W / \partial c^j \partial c^i$, the deformation does not contribute to the polar part. Expanding $\gamma_i(w)$ around z , we obtain

$$J^{(g)}(w)G(z) \sim \frac{2iN}{(w-z)^3} - \frac{2i a^i \gamma_i(z)}{(w-z)^2} - \frac{2i T(z)}{w-z} + \text{reg.} \tag{6.53}$$

Here and below, reg. denotes terms non polar at $w = z$, of the form $\left(\frac{w-z}{\bar{w}-\bar{z}} \right)^k$.

The $T(w)G(z)$ OPE

We have

$$\begin{aligned}
T(w)G(z) &\sim (a^i \partial \gamma_i + \partial b_i \partial c^i) (w) (a^j \partial b_j) (z) \\
&\sim \frac{a^i(w) \partial b_i(z)}{(w-z)^2} + \frac{\partial b_i(w) a^i(z)}{(w-z)^2} + \text{reg.}
\end{aligned} \tag{6.54}$$

Expanding the fields at w around z gives

$$T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.}$$

The additional contributions from the deformed bb OPE are nonpolar and do not contribute to the singular part; the remaining terms vanish by antisymmetry. Therefore,

$$T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.} \tag{6.55}$$

The $T(w)T(z)$ OPE

We write

$$T(w)T(z) \sim \left(a^i \partial \gamma_i + \partial b_i \partial c^i \right) (w) \left(a^j \partial \gamma_j + \partial b_j \partial c^j \right) (z). \quad (6.56)$$

The free contractions give

$$T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z}.$$

The additional terms involving the deformed $\gamma\bar{\gamma}$, γb , and bb OPEs are nonpolar or vanish by antisymmetry. Hence the Virasoro algebra is unchanged:

$$T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg.} \quad (6.57)$$

In particular, the central charge remains $c = 0$.

The $T(w)J^{(g)}(z)$ OPE

Finally,

$$T(w)J^{(g)}(z) \sim \left(a^i \partial \gamma_i + \partial b_i \partial c^i \right) (w) \left(-2i \gamma_j \partial c^j + g a^j \frac{\partial W}{\partial c^j} \right) (z). \quad (6.58)$$

The free part gives the usual transformation of the BRST current, while the additional contractions involving the W -dependent term are either regular or cancel by antisymmetry. Therefore,

$$T(w)J^{(g)}(z) \sim \frac{J^{(g)}(z)}{(w-z)^2} + \frac{\partial J^{(g)}(z)}{w-z} + \text{reg.} \quad (6.59)$$

In addition, the OPEs

$$T(w)\bar{T}(z), \quad G(w)\bar{G}(z), \quad G(w)G(z)$$

are all regular.

We therefore recover the same OPE structure as in the free theory. This shows that the deformation preserves the local conformal invariance of the theory.

6.11 Quantum BRST Operator

Definition 6.11.1. As in the free Abelian BF theory, we define the quantum BRST operator by

$$Q_{\text{quantum}} \Phi(z) := \frac{1}{4\pi} \oint_{\gamma_z} J_{\text{BRST}}^{(g)}(w) \Phi(z), \quad (6.60)$$

where

$$J_{\text{BRST}}^{(g)} = dw J^{(g)} + d\bar{w} \bar{J}^{(g)},$$

with

$$J^{(g)} = -2i \gamma_i \partial c^i + g a^i \frac{\partial W}{\partial c^i}, \quad \bar{J}^{(g)} = 2i \bar{\gamma}_i \bar{\partial} c^i + g \bar{a}^i \frac{\partial W}{\partial c^i}. \quad (6.61)$$

Proposition 6.11.1. *The quantum BRST operator satisfies the following properties:*

1. Q_{quantum} agrees with the classical BRST differential Q_g on all elementary fields and their derivatives;
2. $Q_{\text{quantum}}^2 = 0$;
3. the stress-energy tensor is Q_{quantum} -exact,

$$Q_{\text{quantum}} G = T. \quad (6.62)$$

Proof. We prove the three statements separately.

1. Using the OPEs of $J^{(g)}$ and $\bar{J}^{(g)}$ with the elementary fields and taking the corresponding contour residues, we obtain

$$\begin{aligned} Q_{\text{quantum}} c^i &= 0, & Q_{\text{quantum}} a^i &= -\partial c^i, & Q_{\text{quantum}} \bar{a}^i &= -\bar{\partial} c^i, \\ Q_{\text{quantum}} \gamma_i &= -\frac{ig}{2} \frac{\partial W}{\partial c^i}, & Q_{\text{quantum}} \bar{\gamma}_i &= \frac{ig}{2} \frac{\partial W}{\partial c^i}, & Q_{\text{quantum}} b_i &= \lambda_i. \end{aligned} \quad (6.63)$$

Here

$$\lambda_i = \gamma_i + \bar{\gamma}_i, \quad Q_{\text{quantum}} \lambda_i = 0.$$

These are precisely the classical BRST transformations generated by Q_g . Since the contour action commutes with derivatives, the same statement holds for all derivatives of the elementary fields.

2. For nilpotence, we use the current-current OPEs

$$J^{(g)}(w) J^{(g)}(z) \sim \frac{4ig}{w-z} \partial W(c(z)), \quad (6.64)$$

and

$$\bar{J}^{(g)}(w)\bar{J}^{(g)}(z) \sim -\frac{4ig}{\bar{w}-\bar{z}} \bar{\partial}W(c(z)). \quad (6.65)$$

The mixed OPEs $J^{(g)}(w)\bar{J}^{(g)}(z)$ and $\bar{J}^{(g)}(w)J^{(g)}(z)$ contain only nonpolar logarithmic terms and therefore have vanishing contour residue. Hence

$$\oint_{\gamma_z} J_{\text{BRST}}^{(g)}(w)J_{\text{BRST}}^{(g)}(z) = -8\pi g dW(c(z)). \quad (6.66)$$

The double contour defining Q_{quantum}^2 is therefore proportional to

$$\oint_{\gamma_z} dW(c(z)) = 0,$$

since dW is an exact one-form. Hence

$$Q_{\text{quantum}}^2 = 0. \quad (6.67)$$

3. Using the OPE between the holomorphic BRST current and G ,

$$J^{(g)}(w)G(z) \sim \frac{2iN}{(w-z)^3} - \frac{2i a^i \gamma_i(z)}{(w-z)^2} - \frac{2i T(z)}{w-z} + \text{reg.}, \quad (6.68)$$

only the simple pole contributes to the contour integral. Therefore

$$\begin{aligned} Q_{\text{quantum}}G(z) &= \frac{1}{4\pi} \oint_{\gamma_z} J_{\text{BRST}}^{(g)}(w)G(z) \\ &= \frac{1}{4\pi} (2\pi i)(-2i) T(z) \\ &= T(z). \end{aligned} \quad (6.69)$$

Thus,

$$Q_{\text{quantum}}G = T. \quad (6.70)$$

□

6.12 Broken R -current Symmetry

Contrary to the free theory, the deformed theory no longer possesses the same twisted superconformal structure [1, Section 2.7]. In particular, the original $U(1)$ current $\gamma_i a^i$ is no longer antiholomorphically conserved.

Assuming that W is homogeneous of degree p , we define a modified R -symmetry

through the phase rotations

$$a^i \mapsto e^{i\alpha} a^i, \quad \gamma_i \mapsto e^{-i\alpha} \gamma_i, \quad c^i \mapsto e^{-i\frac{\alpha}{p-2}} c^i, \quad b_i \mapsto e^{i\frac{\alpha}{p-2}} b_i. \quad (6.71)$$

The corresponding Noether current is

$$J_r = \left(\gamma_i a^i - \frac{1}{p-2} c^i \partial b_i \right) dz + \frac{1}{p-2} b_i \bar{\partial} c^i d\bar{z}. \quad (6.72)$$

Using the equations of motion and the homogeneity of W , one verifies that

$$dJ_r = 0.$$

However, this modified current does not restore the twisted superconformal algebra. Indeed,

$$J_r(w) J^{(g)}(z) \sim -\frac{2i}{p-2} \frac{c^i \gamma_i(z)}{(w-z)^2} + \frac{1}{w-z} \left[\frac{p-1}{p-2} J_0(z) + \cdots \right] + \text{reg.}, \quad (6.73)$$

which is not of the required form

$$J_r(w) J^{(g)}(z) \sim \frac{J^{(g)}(z)}{w-z} + \text{reg.} \quad (6.74)$$

The BRST current does not have the standard R -charge, and the twisted superconformal structure of the free theory is not recovered.

6.13 Deformed Q_g -Cohomology

6.13.1 Local Q_g -Cohomology

As in the free theory, we define the space of composite fields by

$$\mathbb{F} := \frac{\text{Sym}^\bullet \text{Jet}(\underline{F}_{\mathbb{C}}^*)}{\langle \text{equations of motion} \rangle}, \quad (6.75)$$

and denote by $\mathbb{F}_z$ its fiber at $z \in \Sigma$. The polynomial local observables are

$$\mathcal{O}_z := H_{Q_g}^\bullet(\mathbb{F}_z). \quad (6.76)$$

We also consider the distributional extension

$$\mathbb{F}_z \subset \mathbb{F}_z^\delta,$$

obtained by allowing Dirac delta distributions of the elementary fields, their distributional derivatives, and admissible finite products. Its cohomology is

$$\mathcal{O}_z^\delta := H_{Q_g^\delta}^\bullet(\mathbb{F}_z^\delta), \quad (6.77)$$

where Q_g^δ denotes the distributional extension of Q_g .

Free theory. For $g = 0$, we recover the free BRST differential Q_0 . Its polynomial cohomology is

$$\mathcal{O}_{z,0} = H_{Q_0}^\bullet(\mathbb{F}_z) = \mathbb{C}[B_1, \dots, B_N] \otimes \Lambda(c^1, \dots, c^N). \quad (6.78)$$

The extended cohomology is

$$\mathcal{O}_{z,0}^\delta := H_{Q_0}^\bullet(\mathbb{F}_z^\delta).$$

Natural Q_0 -closed distributional representatives include

$$\begin{aligned} \delta(a^i) \partial c^i, \quad \delta(\bar{a}^i) \bar{\partial} c^i, \quad b_i \delta(\lambda_i), \quad \delta^{(n)}(B_i), \\ \delta^{(n)}(\gamma_i), \quad \delta^{(n)}(\bar{\gamma}_i), \quad b_i \delta(\gamma_i) \delta(\bar{\gamma}_i), \quad n \geq 0. \end{aligned} \quad (6.79)$$

Deformed polynomial cohomology. After the deformation,

$$Q_g c^i = 0, \quad Q_g B_i = -g W_i(c), \quad W_i(c) := \frac{\partial W}{\partial c^i}.$$

On the polynomial algebra $\mathbb{C}[B_1, \dots, B_N] \otimes \Lambda(c^1, \dots, c^N)$, the differential reduces to

$$Q_g = -g \sum_{i=1}^N W_i(c) \frac{\partial}{\partial B_i}. \quad (6.80)$$

The polynomial Q_g -cohomology is

$$\mathcal{O}_z = H^\bullet(\mathbb{C}[B_1, \dots, B_N] \otimes \Lambda(c^1, \dots, c^N), Q_g). \quad (6.81)$$

Extended cohomology. The distributional extension of the differential is

$$Q_g^\delta = \sum_{i=1}^N \left(\lambda_i \frac{\partial}{\partial b_i} - \partial c^i \frac{\partial}{\partial a^i} - \bar{\partial} c^i \frac{\partial}{\partial \bar{a}^i} - g W_i(c) \frac{\partial}{\partial B_i} - \frac{ig}{2} W_i(c) \frac{\partial}{\partial \gamma_i} + \frac{ig}{2} W_i(c) \frac{\partial}{\partial \bar{\gamma}_i} \right). \quad (6.82)$$

In particular,

$$Q_g^\delta \delta^{(n)}(B_i) = -g W_i(c) \delta^{(n+1)}(B_i), \quad n \geq 0. \quad (6.83)$$

The fields $\delta(B_i)$, $\delta(\gamma_i)$, $\delta(\bar{\gamma}_i)$ and their derivatives, as well as $b_i \delta(\gamma_i) \delta(\bar{\gamma}_i)$, are therefore no longer Q_g^δ -closed in general. For $g \neq 0$,

$$W_i(c) \delta^{(n)}(B_i) = -\frac{1}{g} Q_g^\delta \delta^{(n-1)}(B_i), \quad n \geq 1, \quad (6.84)$$

so these terms yield no new cohomology classes.

For each i , with no sum over i , the following fields are Q_g^δ -closed:

$$\begin{aligned} \delta(a^i) \partial c^i, \quad \delta(\bar{a}^i) \bar{\partial} c^i, \quad b_i \delta(\lambda_i), \\ W_i(c) \delta(B_i), \quad b_i W_i(c) \delta(\gamma_i) \delta(\bar{\gamma}_i). \end{aligned} \quad (6.85)$$

The closure of the last two representatives uses $W_i(c)^2 = 0$, since $W_i(c)$ is odd.

The extended cohomology is

$$\mathcal{O}_z^\delta = H_{Q_g^\delta}^\bullet(\mathbb{F}_z^\delta), \quad (6.86)$$

and contains, in particular, the classes represented by

$$\delta(a^i) \partial c^i, \quad \delta(\bar{a}^i) \bar{\partial} c^i, \quad b_i \delta(\lambda_i), \quad W_i(c) \delta(B_i),$$

together with their admissible products. For a general W , further relations may arise from relations among the functions $W_i(c)$.

6.14 Special Cases

6.14.1 Quadratic Potential

Non-degenerate Quadratic W . Let

$$W(c) = \frac{1}{2} w_{ij} c^i c^j. \quad (6.87)$$

Then

$$W_i(c) := \frac{\partial W}{\partial c^i} = w_{ij} c^j.$$

If

$$\det w \neq 0,$$

we have

$$Q_g B_i = -g w_{ij} c^j.$$

We can therefore introduce the new variables

$$c'_i := -g w_{ij} c^j.$$

Since w is invertible, this defines an invertible change of variables, and

$$Q_g B_i = c'_i, \quad Q_g c'_i = 0.$$

Thus (B_i, c'_i) form contractible pairs, and consequently

$$\mathcal{O}_z \simeq \mathbb{C}. \tag{6.88}$$

Among the distributional representatives considered above, the classes that remain Q_g^δ -closed are represented by

$$\delta(a^i) \partial c^i, \quad \delta(\bar{a}^i) \bar{\partial} c^i, \quad b_i \delta(\lambda_i), \quad W_i(c) \delta(B_i).$$

They generate a subalgebra of the extended cohomology,

$$\left\langle \delta(a^i) \partial c^i, \delta(\bar{a}^i) \bar{\partial} c^i, b_i \delta(\lambda_i), W_i(c) \delta(B_i) \right\rangle_{\text{alg}, i=1}^N \subseteq \mathcal{O}_z^\delta. \tag{6.89}$$

Degenerate quadratic W . Let

$$W(c) = \frac{1}{2} w_{ij} c^i c^j = \sum_{i < j} w_{ij} c^i c^j. \tag{6.90}$$

Assume that $w \neq 0$. Then there exist $e_1, e_2 \in V$ such that

$$w(e_1, e_2) = m_1 \neq 0.$$

Define

$$U_1 := \text{span}(e_1, e_2).$$

On U_1 , the bilinear form w is represented by

$$w|_{U_1} = \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}.$$

We define

$$U_1^\perp := \{v \in V \mid w(v, u) = 0 \text{ for every } u \in U_1\}.$$

Then

$$V = U_1 \oplus U_1^\perp.$$

If

$$w|_{U_1^\perp} \neq 0,$$

we choose $e_3, e_4 \in U_1^\perp$ such that

$$w(e_3, e_4) = m_2 \neq 0,$$

and continue in the same way. We finally obtain a decomposition

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_r \oplus K, \quad w|_K = 0, \quad (6.91)$$

where

$$U_i = \text{span}(e_{2i-1}, e_{2i}), \quad 1 \leq i \leq r.$$

In this basis, w can be written as

$$w = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

and

$$\text{rank } w = 2r.$$

The quadratic potential takes the form

$$W(c) = \sum_{i=1}^r m_i c^{2i-1} c^{2i}. \quad (6.92)$$

For $i = 1, \dots, r$, we therefore have

$$\frac{\partial W}{\partial c^{2i-1}} = m_i c^{2i}, \quad \frac{\partial W}{\partial c^{2i}} = -m_i c^{2i-1}.$$

Hence

$$Q_g B_{2i-1} = -gm_i c^{2i}, \quad Q_g B_{2i} = gm_i c^{2i-1}.$$

Thus

$$(B_{2i-1}, c^{2i}), \quad (B_{2i}, c^{2i-1}), \quad 1 \leq i \leq r,$$

form contractible pairs. The variables corresponding to the kernel of w remain in the cohomology, and we obtain

$$\mathcal{O}_z \simeq \mathbb{C}[B_{2r+1}, \dots, B_N] \otimes \Lambda(c^{2r+1}, \dots, c^N). \quad (6.93)$$

For the extended cohomology $\mathcal{O}_z^\delta$, additional distributional classes survive. In particular,

$$\delta(a^i)\partial c^i, \quad \delta(\bar{a}^i)\bar{\partial} c^i, \quad b_i\delta(\lambda_i), \quad i = 1, \dots, N,$$

remain Q_g^δ -closed.

Moreover, for the kernel directions

$$j = 2r + 1, \dots, N,$$

we have

$$W_j(c) = 0,$$

and therefore

$$Q_g B_j = 0, \quad Q_g \gamma_j = 0, \quad Q_g \bar{\gamma}_j = 0.$$

The corresponding distributional operators

$$\delta^{(n)}(B_j), \quad \delta^{(n)}(\gamma_j), \quad \delta^{(n)}(\bar{\gamma}_j), \quad b_j\delta(\gamma_j)\delta(\bar{\gamma}_j), \quad n \geq 0,$$

are also Q_g^δ -closed.

Thus, for a degenerate quadratic potential of rank $2r$, the extended cohomology $\mathcal{O}_z^\delta$ contains the polynomial cohomology

$$\mathbb{C}[B_{2r+1}, \dots, B_N] \otimes \Lambda(c^{2r+1}, \dots, c^N),$$

together with the distributional classes represented by

$$\boxed{\begin{array}{llll} \delta(a^i)\partial c^i, & \delta(\bar{a}^i)\bar{\partial} c^i, & b_i\delta(\lambda_i), & i = 1, \dots, N, \\ W_i(c)\delta(B_i), & & & i = 1, \dots, 2r, \\ \delta^{(n)}(B_i), & \delta^{(n)}(\gamma_i), & \delta^{(n)}(\bar{\gamma}_i), & i = 2r + 1, \dots, N, \quad n \geq 0, \\ b_i\delta(\gamma_i)\delta(\bar{\gamma}_i), & & & i = 2r + 1, \dots, N. \end{array}} \quad (6.94)$$

General W with non-degenerate quadratic part. Let W have a non-degenerate quadratic part. Write

$$W_i(c) := \frac{\partial W}{\partial c^i} = w_{ij}c^j + P_i(c), \quad (6.95)$$

where w is non-degenerate and $P_i(c)$ contains terms of degree at least three in the fields c^i . We then have

$$c^j = (w^{-1})^{ji} W_i(c) - (w^{-1})^{ji} P_i(c).$$

Substituting this expression recursively into the higher-order terms yields, after finitely many steps,

$$\Lambda(c^1, \dots, c^N) = \Lambda(W_1(c), \dots, W_N(c)). \quad (6.96)$$

Since

$$Q_g B_i = -g W_i(c), \quad Q_g W_i(c) = 0,$$

the change of variables

$$c'_i := W_i(c)$$

is invertible, and the pairs (B_i, c'_i) are contractible. This yields

$$\mathcal{O}_z \simeq \mathbb{C}. \quad (6.97)$$

As in the non-degenerate quadratic case, among the distributional representatives considered above, the classes that remain Q_g^δ -closed are represented by

$$\delta(a^i) \partial c^i, \quad \delta(\bar{a}^i) \bar{\partial} c^i, \quad b_i \delta(\lambda_i), \quad W_i(c) \delta(B_i), \quad i = 1, \dots, N.$$

They generate a subalgebra of the extended cohomology,

$$\left\langle \delta(a^i) \partial c^i, \delta(\bar{a}^i) \bar{\partial} c^i, b_i \delta(\lambda_i), W_i(c) \delta(B_i) \right\rangle_{\text{alg}, i=1}^N \subseteq \mathcal{O}_z^\delta. \quad (6.98)$$

Chapter 7

2D BF Theory with the Delta Deformation

7.1 δ -interaction

We consider the observable

$$\mathcal{O}^{(0)} = \delta(a) \partial c \delta(\bar{a}) \bar{\partial} c. \quad (7.1)$$

By Witten's descent procedure,

$$\begin{aligned} \mathcal{O}^{(1)} &= \Gamma \mathcal{O}^{(0)} = -dz G_{-1} \mathcal{O}^{(0)} - d\bar{z} \bar{G}_{-1} \mathcal{O}^{(0)} \\ &= dz \delta(a) \partial a \delta(\bar{a}) \bar{\partial} c - d\bar{z} \delta(a) \partial c \delta(\bar{a}) \bar{\partial} \bar{a}, \end{aligned} \quad (7.2)$$

and

$$\begin{aligned} \mathcal{O}^{(2)} &= -dz \wedge d\bar{z} G_{-1} \bar{G}_{-1} \mathcal{O}^{(0)} \\ &= -dz \wedge d\bar{z} \delta(a) \partial a \delta(\bar{a}) \bar{\partial} \bar{a}. \end{aligned} \quad (7.3)$$

The deformed action is

$$\begin{aligned} S_g &= S_0 + g \int_{\Sigma} \mathcal{O}^{(2)} \\ &= 2i \int_{\Sigma} dz \wedge d\bar{z} \left(\gamma \bar{\partial} a + \bar{\gamma} \partial \bar{a} + b \partial \bar{\partial} c + \frac{g}{2i} \delta(a) \partial a \delta(\bar{a}) \bar{\partial} \bar{a} \right). \end{aligned} \quad (7.4)$$

7.2 Equations of Motion

The equations of motion of the deformed theory are

$$\begin{aligned}\bar{\partial}a &= 0, & \partial\bar{a} &= 0, & \partial\bar{\partial}c &= 0, & \partial\bar{\partial}b &= 0, \\ 2i\bar{\partial}\gamma + g\delta(a)\partial(\delta(\bar{a})\bar{\partial}\bar{a}) &= 0, \\ 2i\partial\bar{\gamma} + g\bar{\partial}(\delta(a)\partial a)\delta(\bar{a}) &= 0.\end{aligned}\tag{7.5}$$

7.3 Deformed BRST Operator and Current

We write the deformed BRST differential as

$$Q_g = Q_0 + gQ_1,$$

where Q_0 is the BRST differential of the free Abelian BF theory. From (7.4), we write

$$S_g = S_0 + gS_1, \quad S_1 = \int_{\Sigma} dz \wedge d\bar{z} \delta(a) \partial a \delta(\bar{a}) \bar{\partial}\bar{a}.$$

BRST invariance at first order in g requires

$$Q_0S_1 + Q_1S_0 = 0.\tag{7.6}$$

After integration by parts and discarding boundary terms, we obtain

$$\begin{aligned}Q_0S_1 &= \int_{\Sigma} dz \wedge d\bar{z} \left[\delta(\bar{a}) \bar{\partial}(\delta(a)\partial c) \partial\bar{a} \right. \\ &\quad \left. - \delta(a) \partial(\delta(\bar{a})\bar{\partial}c) \bar{\partial}a \right].\end{aligned}\tag{7.7}$$

On the other hand,

$$Q_1S_0 = 2i \int_{\Sigma} dz \wedge d\bar{z} \left[(Q_1\gamma)\bar{\partial}a + (Q_1\bar{\gamma})\partial\bar{a} \right].$$

The contribution in (7.7) is canceled by choosing

$$Q_1\gamma = \frac{1}{2i} \delta(a) \partial(\delta(\bar{a})\bar{\partial}c), \quad Q_1\bar{\gamma} = -\frac{1}{2i} \delta(\bar{a}) \bar{\partial}(\delta(a)\partial c).\tag{7.8}$$

The remaining BRST transformations are unchanged:

$$\begin{aligned}Q_g a &= -\partial c, & Q_g \bar{a} &= -\bar{\partial}c, & Q_g c &= 0, \\ Q_g b &= \gamma + \bar{\gamma}, & Q_g \gamma &= \frac{g}{2i} \delta(a) \partial(\delta(\bar{a})\bar{\partial}c), & Q_g \bar{\gamma} &= -\frac{g}{2i} \delta(\bar{a}) \bar{\partial}(\delta(a)\partial c).\end{aligned}\tag{7.9}$$

Using the equations of motion

$$\bar{\partial}a = 0, \quad \partial\bar{a} = 0, \quad \partial\bar{\partial}c = 0,$$

we obtain

$$Q_g \gamma \underset{\text{EOM}}{\sim} 0, \quad Q_g \bar{\gamma} \underset{\text{EOM}}{\sim} 0.$$

Hence

$$Q_g \underset{\text{EOM}}{\sim} Q_0,$$

which yields

$$Q_g^2 \underset{\text{EOM}}{\sim} 0. \quad (7.10)$$

Thus, the deformed BRST differential Q_g is nilpotent only on shell.

It follows immediately that, on shell, the BRST current of the deformed theory is equivalent to the free BRST current:

$$J_{\text{BRST}}^{(g)} \underset{\text{EOM}}{\sim} J_{\text{BRST}}^{(0)} = 2i \left(-dz \gamma \partial c + d\bar{z} \bar{\gamma} \bar{\partial} c \right). \quad (7.11)$$

7.4 Stress-Energy Tensor and OPEs

In the complex-coordinate formulation, the explicit dependence on the metric is no longer visible. In two dimensions, Theorem 2.2.2 allows us to define the stress-energy tensor by deforming the complex structure.

Using the Beltrami differential μ , we deform the holomorphic and antiholomorphic Dolbeault operators as

$$\bar{\partial}_\mu = \bar{\partial} + \mu \partial, \quad \partial_\mu = \partial + \bar{\mu} \bar{\partial}. \quad (7.12)$$

The corresponding basis of one-forms is

$$dz_\mu = dz - \mu d\bar{z}, \quad d\bar{z}_{\bar{\mu}} = d\bar{z} - \bar{\mu} dz. \quad (7.13)$$

The gauge field A decomposes in this basis as

$$\begin{aligned} A &= a_\mu dz_\mu + a_{\bar{\mu}} d\bar{z}_{\bar{\mu}} \\ &= a_\mu (dz - \mu d\bar{z}) + a_{\bar{\mu}} (d\bar{z} - \bar{\mu} dz) \\ &= (a_\mu - \bar{\mu} a_{\bar{\mu}}) dz + (a_{\bar{\mu}} - \mu a_\mu) d\bar{z} \\ &= a dz + \bar{a} d\bar{z}. \end{aligned} \quad (7.14)$$

Therefore,

$$a_\mu = a + \bar{\mu} a_{\bar{\mu}}, \quad a_{\bar{\mu}} = \bar{a} + \mu a_\mu. \quad (7.15)$$

To first order in the Beltrami differential,

$$dz_\mu \wedge d\bar{z}_{\bar{\mu}} = dz \wedge d\bar{z} + \mathcal{O}(\mu\bar{\mu}). \quad (7.16)$$

We define the stress-energy tensor by

$$\delta_\mu S_g = -2i \int_\Sigma dz \wedge d\bar{z} \left(\mu T + \bar{\mu} \bar{T} \right). \quad (7.17)$$

For the delta deformation, setting $\bar{\mu} = 0$ gives

$$\begin{aligned} T &= \partial b \partial c + a \partial \gamma + \frac{g}{2i} \bar{\partial}(\delta(a) \partial a \delta(\bar{a}) a) \\ &\quad - \frac{g}{2i} \delta(a) \partial a \delta(\bar{a}) (\bar{\partial} a + \partial \bar{a}) a \delta'(a) \bar{\partial} \bar{a}. \end{aligned} \quad (7.18)$$

Using the distributional identities

$$a \delta(a) = 0, \quad a \delta'(a) = -\delta(a),$$

the deformation terms cancel. Hence

$$T = \partial b \partial c + a \partial \gamma, \quad \bar{T} = \bar{\partial} b \bar{\partial} c + \bar{a} \bar{\partial} \bar{\gamma}. \quad (7.19)$$

Since

$$Q_g \underset{\text{EOM}}{\sim} Q_0,$$

we obtain

$$T \underset{\text{EOM}}{\sim} Q_g(G), \quad \bar{T} \underset{\text{EOM}}{\sim} Q_g(\bar{G}), \quad (7.20)$$

with

$$G = a \partial b, \quad \bar{G} = \bar{a} \bar{\partial} b. \quad (7.21)$$

The corresponding OPEs take the same form as in the free theory:

$$T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg.}, \quad (7.22)$$

$$J^{(g)}(w)G(z) \sim -\frac{1}{(w-z)^3} + \frac{(\gamma a)(z)}{(w-z)^2} + \frac{T(z)}{w-z} + \text{reg.}, \quad (7.23)$$

$$T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.}, \quad (7.24)$$

$$T(w)J^{(g)}(z) \sim \frac{J^{(g)}(z)}{(w-z)^2} + \frac{\partial J^{(g)}(z)}{w-z} + \text{reg.} \quad (7.25)$$

7.5 BV Completion

To handle the open BRST algebra off shell, we extend the theory to the Batalin–Vilkovisky formalism and check the failure of the classical master equation

$$\{S, S\} = 0.$$

We start from the linear antifield extension of the deformed action S_g :

$$S_{\text{BV}}^{\text{lin}} := S_g + S_{\text{lin}}, \quad S_{\text{lin}} := \int_{\Sigma} dz \wedge d\bar{z} (Q_g \Phi^A) \Phi_A^+. \quad (7.26)$$

Explicitly,

$$S_{\text{lin}} = \int_{\Sigma} dz \wedge d\bar{z} \left[(Q_g \gamma) \gamma^+ + (Q_g \bar{\gamma}) \bar{\gamma}^+ - (\partial c) a^+ - (\bar{\partial} c) \bar{a}^+ + (\gamma + \bar{\gamma}) b^+ \right]. \quad (7.27)$$

The antibracket of $S_{\text{BV}}^{\text{lin}}$ with itself is

$$\{S_{\text{BV}}^{\text{lin}}, S_{\text{BV}}^{\text{lin}}\} = \{S_g, S_g\} + 2\{S_g, S_{\text{lin}}\} + \{S_{\text{lin}}, S_{\text{lin}}\}. \quad (7.28)$$

Since S_g depends only on the fields, $\{S_g, S_g\} = 0$. BRST invariance, $Q_g S_g = 0$, also gives $\{S_g, S_{\text{lin}}\} = -Q_g S_g = 0$. The remaining term is

$$\{S_{\text{lin}}, S_{\text{lin}}\} = -2 \int_{\Sigma} dz \wedge d\bar{z} \left[(Q_g^2 \gamma) \gamma^+ + (Q_g^2 \bar{\gamma}) \bar{\gamma}^+ + (Q_g^2 b) b^+ \right]. \quad (7.29)$$

It does not vanish off shell because Q_g is not off-shell nilpotent.

We define the coefficients and equations of motion by

$$\begin{aligned}
A &:= \delta'(a)\delta'(\bar{a})\partial c\bar{\partial}c, & B &:= \delta'(a)\delta(\bar{a})\partial c, & E_\gamma &:= \frac{\delta S_g}{\delta\gamma} = 2i\bar{\partial}a, \\
C &:= \delta'(\bar{a})\delta(a)\bar{\partial}c, & D &:= \delta(a)\delta(\bar{a}), & E_{\bar{\gamma}} &:= \frac{\delta S_g}{\delta\bar{\gamma}} = 2i\partial\bar{a}, \\
& & & & E_b &:= \frac{\delta S_g}{\delta b} = 2i\partial\bar{\partial}c.
\end{aligned} \tag{7.30}$$

The failure of nilpotency is proportional to the equations of motion:

$$\begin{pmatrix} Q_g^2\gamma \\ Q_g^2\bar{\gamma} \\ Q_g^2b \end{pmatrix} = \frac{g}{4} \begin{pmatrix} 0 & A & B \\ -A & 0 & -C \\ -B & C & -2D \end{pmatrix} \begin{pmatrix} E_\gamma \\ E_{\bar{\gamma}} \\ E_b \end{pmatrix}, \tag{7.31}$$

while

$$Q_g^2a = Q_g^2\bar{a} = Q_g^2c = 0.$$

Using

$$\{S_g, \gamma^+\} = E_\gamma, \quad \{S_g, \bar{\gamma}^+\} = E_{\bar{\gamma}}, \quad \{S_g, b^+\} = -E_b,$$

Following the BV completion mechanism used in the Poisson sigma model [16], we introduce the quadratic antifield term

$$\begin{aligned}
S_{\text{quad}} &= -\frac{1}{2} \int_\Sigma dz \wedge d\bar{z} \begin{pmatrix} \gamma^+ & \bar{\gamma}^+ & b^+ \end{pmatrix} \frac{g}{4} \begin{pmatrix} 0 & A & B \\ -A & 0 & -C \\ -B & C & -2D \end{pmatrix} \begin{pmatrix} \gamma^+ \\ \bar{\gamma}^+ \\ b^+ \end{pmatrix} \\
&= -\frac{g}{4} \int_\Sigma dz \wedge d\bar{z} \left[A\gamma^+\bar{\gamma}^+ - B\gamma^+b^+ + C\bar{\gamma}^+b^+ - D(b^+)^2 \right].
\end{aligned} \tag{7.32}$$

By construction,

$$\frac{1}{2}\{S_{\text{lin}}, S_{\text{lin}}\} + \{S_g, S_{\text{quad}}\} = 0. \tag{7.33}$$

For the complete action

$$S_{\text{BV}} := S_g + S_{\text{lin}} + S_{\text{quad}},$$

a direct calculation gives

$$\{S_{\text{lin}}, S_{\text{quad}}\} = 0, \quad \{S_{\text{quad}}, S_{\text{quad}}\} = 0.$$

The classical master equation therefore holds off shell:

$$\{S_{\text{BV}}, S_{\text{BV}}\} = 0. \tag{7.34}$$

Setting all antifields to zero yields S_g , confirming the off-shell BV consistency of the deformed theory.

7.6 Correlators

Example 7.6.1 (First Correlator on $\mathbb{C}$). We first calculate the correlator

$$\langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g$$

on $\mathbb{C}$. We have

$$\begin{aligned} & \langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g \\ &= -\frac{g}{4\pi} \int_{\mathbb{C}} d^2w \left\langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \delta(a(w)) \partial a(w) \delta(\bar{a}(w)) \bar{\partial} \bar{a}(w) \right\rangle_0 \\ &= \frac{g}{4\pi} \int_{\mathbb{C}} d^2w \left\langle \delta(a(w)) \delta(\gamma(z_2)) \partial a(w) \gamma(z_1) \right\rangle_0 \\ & \quad \times \left\langle \delta(\bar{a}(w)) \delta(\bar{\gamma}(z_4)) \bar{\partial} \bar{a}(w) \bar{\gamma}(z_3) \right\rangle_0. \end{aligned} \quad (7.35)$$

Using (4.15) from Section 4 and differentiating with respect to w , we obtain

$$\langle \delta(a(w)) \delta(\gamma(z_2)) \partial a(w) \gamma(z_1) \rangle_0 = -\frac{w - z_2}{(w - z_1)^2} + \frac{1}{w - z_1}. \quad (7.36)$$

Similarly, for the antiholomorphic part,

$$\langle \delta(\bar{a}(w)) \delta(\bar{\gamma}(z_4)) \bar{\partial} \bar{a}(w) \bar{\gamma}(z_3) \rangle_0 = -\frac{\bar{w} - \bar{z}_4}{(\bar{w} - \bar{z}_3)^2} + \frac{1}{\bar{w} - \bar{z}_3}. \quad (7.37)$$

Therefore,

$$\begin{aligned} & \langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g \\ &= \frac{g}{4\pi} (z_2 - z_1)(\bar{z}_4 - \bar{z}_3) \int_{\mathbb{C}} \frac{d^2w}{(w - z_1)^2(\bar{w} - \bar{z}_3)^2}. \end{aligned} \quad (7.38)$$

This integral can be calculated from

$$I = \int_{D_R} \frac{d^2w}{\pi} \frac{1}{(w - z_1)(\bar{w} - \bar{z}_3)} = \log \left(\frac{R^2 - z_1 \bar{z}_3}{|z_1 - z_3|^2} \right), \quad (7.39)$$

where D_R is the disk of radius R . Differentiating with respect to z_1 and $\bar{z}_3$, we find

$$\frac{1}{\pi} \int_{D_R} \frac{d^2w}{(w - z_1)^2(\bar{w} - \bar{z}_3)^2} = \partial_{\bar{z}_3} \partial_{z_1} I = -\frac{R^2}{(R^2 - z_1 \bar{z}_3)^2} + \pi \delta^{(2)}(z_1 - z_3). \quad (7.40)$$

Taking $R \rightarrow \infty$, we obtain

$$\begin{aligned} & \langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g \\ &= \frac{g\pi}{4} (z_2 - z_1)(\bar{z}_4 - \bar{z}_3) \delta^{(2)}(z_1 - z_3). \end{aligned} \quad (7.41)$$

The correlator vanishes for $z_1 \neq z_3$ and is purely a contact term on the collision locus $z_1 = z_3 = z$. To examine the collision $z_1 = z_3 = z$, we return to the integral representation:

$$\begin{aligned} & \langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g \\ &= \frac{g}{4\pi} (z_2 - z)(\bar{z}_4 - \bar{z}) \int_{\mathbb{C}} \frac{d^2 w}{|w - z|^4}. \end{aligned} \quad (7.42)$$

Introducing a cutoff by removing the disk

$$D_\varepsilon(z) := \{w \in \mathbb{C} \mid |w - z| < \varepsilon\},$$

we find

$$\int_{\mathbb{C} \setminus D_\varepsilon(z)} \frac{d^2 w}{|w - z|^4} \sim \frac{\pi}{\varepsilon^2}.$$

Hence,

$$\langle \gamma(z_1) \delta(\gamma(z_2)) \bar{\gamma}(z_3) \delta(\bar{\gamma}(z_4)) \rangle_g \sim \frac{1}{\varepsilon^2},$$

which shows that the collision produces a quadratic ultraviolet divergence.

To further investigate the effect of this deformation, we compute another correlator, this time on the sphere $\mathbb{CP}^1$.

Example 7.6.2 (Correlator on $\mathbb{CP}^1$). We compute the following correlator:

$$\begin{aligned} & \langle \delta(\gamma(z_1)) \delta(\gamma(z_2)) \gamma(z_3) \delta(\bar{\gamma}(z_4)) \delta(\bar{\gamma}(z_5)) \bar{\gamma}(z_6) \rangle_g \\ &= -\frac{g}{4\pi} \int_{\mathbb{CP}^1} d^2 w \langle \delta(\gamma(z_1)) \delta(\gamma(z_2)) \gamma(z_3) \partial a(w) \delta(a(w)) \rangle_0 \\ & \quad \times \langle \delta(\bar{\gamma}(z_4)) \delta(\bar{\gamma}(z_5)) \bar{\gamma}(z_6) \delta(\bar{a}(w)) \bar{\partial} \bar{a}(w) \rangle_0. \end{aligned} \quad (7.43)$$

Using (4.29), the holomorphic correlator is

$$\langle \delta(\gamma(z_1)) \delta(\gamma(z_2)) \gamma(z_3) \partial a(w) \delta(a(w)) \rangle_0 = \frac{(z_3 - z_1)(z_3 - z_2)}{(z_2 - z_1)(w - z_3)^2}. \quad (7.44)$$

The same formula applies to the antiholomorphic sector:

$$\langle \delta(\bar{\gamma}(z_4)) \delta(\bar{\gamma}(z_5)) \bar{\gamma}(z_6) \delta(\bar{a}(w)) \bar{\partial} \bar{a}(w) \rangle_0 = \frac{(\bar{z}_6 - \bar{z}_4)(\bar{z}_6 - \bar{z}_5)}{(\bar{z}_5 - \bar{z}_4)(\bar{w} - \bar{z}_6)^2}. \quad (7.45)$$

We therefore obtain

$$\begin{aligned}
& \langle \delta(\gamma(z_1)) \delta(\gamma(z_2)) \gamma(z_3) \delta(\bar{\gamma}(z_4)) \delta(\bar{\gamma}(z_5)) \bar{\gamma}(z_6) \rangle_g \\
&= -\frac{g}{4\pi} \frac{(z_3 - z_1)(z_3 - z_2)(\bar{z}_6 - \bar{z}_4)(\bar{z}_6 - \bar{z}_5)}{(z_2 - z_1)(\bar{z}_5 - \bar{z}_4)} \int_{\mathbb{CP}^1} \frac{d^2 w}{(w - z_3)^2 (\bar{w} - \bar{z}_6)^2} \\
&= -\frac{g\pi}{4} \frac{(z_3 - z_1)(z_3 - z_2)(\bar{z}_6 - \bar{z}_4)(\bar{z}_6 - \bar{z}_5)}{(z_2 - z_1)(\bar{z}_5 - \bar{z}_4)} \delta^{(2)}(z_3 - z_6). \tag{7.46}
\end{aligned}$$

This correlator is also a pure contact term, now supported on the collision locus $z_3 = z_6$. At $\mathcal{O}(g)$, the corrections to both correlators therefore vanish whenever the insertion points are pairwise distinct; the effect of the deformation in these examples is localized entirely on the diagonals of the configuration space.

These two examples are insufficient to conclude that all first- and higher-order corrections generated by the deformation are strictly contact-like.

Example 7.6.3 (Second Correlator on $\mathbb{C}$). We consider the correlator

$$\langle \delta(B(z_1)) \delta(B(z_2)) B(z_3) B(z_4) \rangle_g.$$

Using Witten's formalism, we find

$$\langle \delta(B(z_1)) \delta(B(z_2)) B(z_3) B(z_4) \rangle_g = \int_{\mathbb{C}} d^2 u G(u; z_1, z_2, z_3, z_4). \tag{7.47}$$

We introduce

$$D(u; z_1, z_2) := \frac{1}{(u - z_1)(\bar{u} - \bar{z}_2)} - \frac{1}{(u - z_2)(\bar{u} - \bar{z}_1)}. \tag{7.48}$$

For $z_i = z_3$ or z_4 , define

$$\begin{aligned}
N(u; z_1, z_2, z_i) &:= \frac{z_1 - z_2}{(u - z_1)^2 (u - z_2)^2 (\bar{u} - \bar{z}_i)} - \frac{z_1 - z_i}{(u - z_1)^2 (u - z_i)^2 (\bar{u} - \bar{z}_2)} \\
&\quad + \frac{z_2 - z_i}{(u - z_2)^2 (u - z_i)^2 (\bar{u} - \bar{z}_1)}, \tag{7.49}
\end{aligned}$$

$$\begin{aligned}
\bar{N}(u; z_1, z_2, z_i) &:= \frac{\bar{z}_2 - \bar{z}_i}{(u - z_1)(\bar{u} - \bar{z}_2)^2 (\bar{u} - \bar{z}_i)^2} - \frac{\bar{z}_1 - \bar{z}_i}{(u - z_2)(\bar{u} - \bar{z}_1)^2 (\bar{u} - \bar{z}_i)^2} \\
&\quad + \frac{\bar{z}_1 - \bar{z}_2}{(u - z_i)(\bar{u} - \bar{z}_1)^2 (\bar{u} - \bar{z}_2)^2}. \tag{7.50}
\end{aligned}$$

The integrand is

$$G(u; z_1, z_2, z_3, z_4) = -\frac{N(u; z_1, z_2, z_3) \bar{N}(u; z_1, z_2, z_4) + N(u; z_1, z_2, z_4) \bar{N}(u; z_1, z_2, z_3)}{D(u; z_1, z_2)^3}. \tag{7.51}$$

Numerically, G exhibits local singularities, together with a non-local singularity when u , z_1 , and z_2 are collinear. These singularities make the integral divergent, so a suitable regularization prescription is required to assign it a finite value.

7.7 Discussion and Results

At first sight, the deformation might appear trivial at first order after examining some of its correlators. However, from (5.98), we already know that $\delta(a)\partial c$ and $\delta(\bar{a})\bar{\partial}c$ are not Q -exact.

We now consider in the free theory

$$\left\langle \left(\delta(a)\partial c \delta(\bar{a})\bar{\partial}c \right)(z_1) b\delta(\lambda)(z_2) b\delta(\lambda)(z_3) \right\rangle_{0,\mathbb{C}}.$$

Up to an overall normalization,

$$\begin{aligned} & \left\langle \left(\delta(a)\partial c \delta(\bar{a})\bar{\partial}c \right)(z_1) b\delta(\lambda)(z_2) b\delta(\lambda)(z_3) \right\rangle_{0,\mathbb{C}} \\ &= \langle \delta(a)(z_1)\delta(\bar{a})(z_1)\delta(\lambda)(z_2)\delta(\lambda)(z_3) \rangle_0 \left\langle \partial c(z_1)\bar{\partial}c(z_1)b(z_2)b(z_3) \right\rangle_0. \end{aligned} \quad (7.52)$$

By (4.13),

$$\langle \delta(a)(z_1)\delta(\bar{a})(z_1)\delta(\lambda)(z_2)\delta(\lambda)(z_3) \rangle_0 = \frac{|z_1 - z_2|^2 |z_1 - z_3|^2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_3) - (z_1 - z_3)(\bar{z}_1 - \bar{z}_2)}. \quad (7.53)$$

The ghost-sector correlator is

$$\left\langle \partial c(z_1)\bar{\partial}c(z_1)b(z_2)b(z_3) \right\rangle_0 = \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_3) - (z_1 - z_3)(\bar{z}_1 - \bar{z}_2)}{|z_1 - z_2|^2 |z_1 - z_3|^2}. \quad (7.54)$$

The two factors cancel, yielding

$$\left\langle \left(\delta(a)\partial c \delta(\bar{a})\bar{\partial}c \right)(z_1) b\delta(\lambda)(z_2) b\delta(\lambda)(z_3) \right\rangle_{0,\mathbb{C}} = 1, \quad (7.55)$$

after fixing the overall normalization. Under the same assumptions as in Proposition 5.11.1, this nonvanishing correlator shows that

$$\delta(a)\partial c \delta(\bar{a})\bar{\partial}c$$

is not Q -exact. The deformation is therefore already cohomologically nontrivial at first order, even though some of its perturbative corrections reduce to contact terms.

A different picture emerges if the algebra of distributional observables is enlarged

further by allowing Heaviside distributions

$$\Theta(a), \quad \Theta(\bar{a}).$$

We have,

$$\begin{aligned} d(\Theta(a) \delta(\bar{a}) d\bar{a}) &= \delta(a) \delta(\bar{a}) da \wedge d\bar{a} \\ &= \delta(a) \delta(\bar{a}) \left(\partial a \bar{\partial} \bar{a} - \bar{\partial} a \partial \bar{a} \right) dz \wedge d\bar{z}. \end{aligned} \quad (7.56)$$

Hence

$$\begin{aligned} \delta(a) \delta(\bar{a}) \partial a \bar{\partial} \bar{a} dz \wedge d\bar{z} \\ = \delta(a) \delta(\bar{a}) \bar{\partial} a \partial \bar{a} dz \wedge d\bar{z} + d(\Theta(a) \delta(\bar{a}) d\bar{a}). \end{aligned} \quad (7.57)$$

This yields:

$$\int_{\Sigma} dz \wedge d\bar{z} \delta(a) \partial a \delta(\bar{a}) \bar{\partial} \bar{a} = \int_{\Sigma} dz \wedge d\bar{z} \delta(a) \bar{\partial} a \delta(\bar{a}) \partial \bar{a}. \quad (7.58)$$

Using

$$\bar{\partial} a \underset{\text{EOM}}{\sim} 0, \quad \partial \bar{a} \underset{\text{EOM}}{\sim} 0,$$

the integrated deformation would then become trivial on shell.

The Heaviside extension also changes the local Q -cohomology. Since $Qa = -\partial c$,

$$Q\Theta(a) = \delta(a) Qa = -\delta(a) \partial c,$$

and therefore

$$\delta(a) \partial c = -Q\Theta(a).$$

Thus a class which is nontrivial in the delta-function extension becomes Q -exact once $\Theta(a)$ is admitted. This is incompatible with the nonvanishing-correlator argument above if the TCFT structure is assumed to extend unchanged to the enlarged algebra. The natural conclusion is not that the deformation is trivial, but rather that the extension from delta distributions to Heaviside distributions requires additional care: the TCFT structure, the treatment of total derivatives, and possible boundary or contact contributions may no longer behave as in the delta-function extension.

7.8 Conclusion

The correlators calculated give several concrete results. At first order, the deformation produces nontrivial distributional Q -cohomology classes; some correlators reduce to pure contact terms, while others develop nonlocal singular loci and require a prescription for their integration. These effects are absent in the undeformed free theory and show that the delta deformation has genuine consequences already at first order. At the same time, the comparison with the Heaviside extension identifies a precise limitation of the distributional TCFT framework used here: delta-function observables are compatible with the calculations performed above, whereas extending the observable algebra to Heaviside distributions remains an open question.

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